A tractable multi-factor dynamic term-structure model for risk management

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Overview

1. The dynamic term-structure model

2. Fitted curves

3. Principal components
   - Definition of principal components
   - Pseudo principal components

4. Pricing
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Two motivations for two purposes

1. **Statistical analysis:** Yield curve movements can be “explained” by only a few factors / principal components. Linear structure necessary for stable PCs ⇒ model yield curve by “deterministic” + “few factors”, e.g. inst. forward rates

\[ f(t, t + \tau) := A(\tau) + M^\top(\tau)u(t), \]

- \( A(\tau) \) deterministic function of time to maturity \( \tau > 0 \)
- \( u(t) \in \mathbb{R}^n \) factor weights at time \( t \)
- \( M(\tau) \in \mathbb{R}^n \) basis functions

2. **Pricing derivatives:** Need no-arbitrage ⇒ choose \( M \) as exponential: Assume \( A = 0 \) and deterministic market. Then

\[ f(t, t + \tau + \Delta) = M^\top(\tau + \Delta)u(t) \overset{!}{=} M^\top(\tau)u(t + \Delta) = f(t + \Delta, t + \Delta + \tau) \]

One solution is \( M_i(\tau) = \exp(-\alpha_i \tau) \), then \( M_i(\tau + \Delta) = M_i(\Delta)M_i(\tau) \).
The model at a glance

- Instantaneous forward rates, seen from time $t$ for the maturity $t + \tau$ modeled as

$$f(t, t + \tau) := A(\tau) + u_1(t)e^{-\alpha_1\tau} + \ldots + u_n(t)e^{-\alpha_n\tau}$$  \hspace{1cm} (1)

- Fixed deterministic exponents $\alpha_n > \ldots > \alpha_1 > 0$

- Deterministic affine term $A(\tau)$

- Stochastic weights $u(t) = (u_1(t), \ldots, u_n(t))^\top$ follow the mean-reverting process

$$du(t) = (\text{diag}(-\alpha)u(t) + R)dt + \sigma dW_t,$$ \hspace{1cm} (2)

where $\sigma \in \mathbb{R}^{n \times n}$, $R \in \mathbb{R}^n$, and $W$ an $n$-dimensional Brownian motion under the real-world measure.
Why...?

- Why this model? Very straightforward linear specification (easiest arbitrage-free affine model)
- Why linear combination of exponentials? Under any (affine-)linear specification of rates and SDE, the function basis $M$ has to be an exponential one because of no-arbitrage conditions
- Why dynamics under the physical measure? Because we see it from a risk management perspective – if you don’t then ignore the risk premium term $R$
- Why $\text{diag}(-\alpha)$ in dynamics of weights $u$? To ease presentation – actually $\text{diag}(-\alpha)$ could be replaced by any matrix $S$, implying an affine-linear risk premium
The dynamic term-structure model

Properties of the model

- Free of arbitrage if for some $b_0$, the affine term is
  \[ A(\tau) = A(0) - \tilde{M}^\top(\tau)(\sigma b_0 - R) - \frac{1}{2} \tilde{M}^\top(\tau)\sigma\sigma^\top \tilde{M}(\tau) \]
  for \[ \tilde{M}_i(\tau) = \int_0^\tau M_i(\tau)\,ds \]
- Affine model in Heath-Jarrow-Morton framework in the sense of [Duffie and Kan(1996)]
- HJM representation of the class of Gaussian short-rate models:
  - Short-rate models: Special case for $n = 1$ Vasicek, case $n = 2$ is known as Hull-White 2-factor or G2++ [Brigo and Mercurio(2006)]
  - HJM representation for arbitrary $n$ theoretically derived in [El Karoui and Lacoste(1995)]
- Gaussian model $\Rightarrow$ easy to simulate, closed-form solutions, well-behaved
- Negative rates possible! (Disadvantage?)
A normal model: Distributional properties

The SDE (2) has an explicit known solution (multi-dimensional Ornstein-Uhlenbeck).

Starting in $t = t_0$, the weights $u(T)$ at time $T > t_0$ are normally distributed with (for $S = \text{diag}(-\alpha)$)

$$u(T) \sim \mathcal{N}(\exp(S(T - t_0))u(t_0) + S^{-1}(\exp(S(T - t_0)) - 1)R, \Sigma_{t_0, T})$$

where the covariance matrix is given by

$$(\Sigma_{t_0, T})_{jk} = -\frac{\exp(- (\alpha_j + \alpha_k)(T - t_0)) - 1}{\alpha_j + \alpha_k}(\sigma\sigma^\top)_{jk}.$$  

Therefore, all (inst.) forward and zero rates are normal too (as linear combinations of normal distribution)!
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Some calibrated curves: Pre financial crisis

All curves displayed were fitted to quotes of German government bonds using a historical calibration. \( n = 5 \) stochastic factors were chosen with an exponent \( \alpha = \frac{1}{12} (1, 2, 3, 4, 5) \top \).

**Figure:** Fitted zero curve on 10/31/2005 (left), and quoted and implied yields for this date (right). Smooth market data result in a very good fit.
Some calibrated curves: Financial crisis

Figure: Fitted zero curve on 10/31/2008 (left), and quoted and implied yields for this date (right). $n = 5$ parameters are not sufficient for a perfect fit, but implied yields are still reasonably close to quoted yields.
Calibrated curves: Error over time

**Figure:** Root mean squared error between quoted and implied yields from 2002 to 2012, for the model presented with $n = 5$ stochastic parameters (blue solid line), with $n = 2$ (green dashed line), and for the benchmark NSS parameterization (red dashed line).
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Recalling principal components

- We are given a statistical covariance estimate \( \hat{\Sigma} \)
- The eigendecomposition \( \hat{\Sigma} = \Gamma \Lambda \Gamma^\top \) with \( \Gamma^\top \Gamma = I \), for \( \Lambda = \text{diag}(\lambda_i) \) with \( \lambda_1 \geq \ldots \geq \lambda_d \) yields the principal components: If a random vector \( X \) has covariance matrix \( \hat{\Sigma} \), then
  \[
  \text{Cov}(\Gamma^\top X) = \Gamma^\top \hat{\Sigma} \Gamma = \Lambda,
  \]
  i.e., the random variables \((\Gamma^\top X)_i\) are uncorrelated with each other
- The columns of \( \Gamma \) as normed eigenvectors are called **principal components**
- If the vector \( X \) is composed of rates with different tenors, then the principal components can be nicely plotted in time
- Typically the largest three eigenvalues account for \( > 95\% \) of yield curve movements as measured by the proportion of “variability”
  \[
  \sum_{i=1}^{3} \lambda_i / \sum_{i=1}^{d} \lambda_i
  \]
Principal components: problems for our model

But:

- Depends on chosen discretization! Although typically tenors are regularly spaced, this arbitrary choice appears nowhere in an explicit fashion!
- Discretizing not necessary for a parametric model where rates are continuously defined
- Discretizing a yield curve on the interval \([0, \infty)\): have to cut somewhere

General solution

Consider a weighted function space \(L^2(\rho)\) on \([0, \infty)\).
Let \( \rho : [0, \infty) \to \mathbb{R}^+_0 \) be a weighting function defining the weighted Hilbert space \( L^2(\rho) \) together with \( \langle f, g \rangle_{L^2(\rho)} := \int_0^{\infty} f(s)g(s)\rho(s)\,ds \).

**Idea**

Instead of operating on the rates, transform the “monomials” \( M \) directly. We represent the result of the transformation by \( P(\tau) = \Gamma^\top M(\tau) \) for \( \Gamma \in \mathbb{R}^{n \times n} \).

The weights wrt \( P(\tau) \) denoted by \( w(t) := \Gamma^{-1}u(t) \) should be uncorrelated, i.e., have a diagonal covariance matrix:

\[
\Lambda^M := \Gamma^{-1}\Sigma\Gamma^{-\top}
\]  

where \( \Sigma = \sigma\sigma^\top \).

(16)
PCs in the yield curve space $L^2(\rho)$ (II)

We require orthonormality in $L^2(\rho)$, i.e. $\langle P_i, P_j \rangle_{L^2(\rho)} = \delta_{ij}$ for $i, j = 1, \ldots, n$ or

$$\Gamma^\top \langle M, M^\top \rangle_{L^2(\rho)} \Gamma = I_n, \quad (6)$$

where $\langle M, M^\top \rangle_{L^2(\rho)} := (\langle M_i, M_j \rangle_{L^2(\rho)})_{i,j} \in \mathbb{R}^{n \times n}$. In this case, for a diagonal matrix $\Lambda^M = \text{diag}(\lambda_i) = \Gamma^{-1} \Sigma \Gamma^{-\top}$,

$$\text{Cov}(df(t, t + \tau_1), df(t, t + \tau_2)) = \text{Cov}(M^\top(\tau_1) du(t), M^\top(\tau_2) du(t))$$

$$= dt \cdot M^\top(\tau_1) \Sigma M(\tau_2)$$

$$= dt \cdot M^\top(\tau_1) \Gamma \Gamma^{-1} \Sigma \Gamma^{-\top} \Gamma^\top M(\tau_2)$$

$$= dt \cdot M^\top(\tau_1) \Gamma \Lambda^M \Gamma^\top M(\tau_2)$$

$$= dt \cdot \sum_{i=1}^{n} \lambda_i (\Gamma^\top M(\tau_1))_i (\Gamma^\top M(\tau_2))_i \quad (7)$$

The instantaneous forward rate covariance is preserved and can be represented by the principal component polynomials $P_i(\tau) = (\Gamma^\top M(\tau))_i$. 
**PCs in the yield curve space $L^2(\rho)$: Back to $\mathbb{R}^n$**

1. Let $D \in \mathbb{R}^{n \times n}$ be a matrix with $DD^\top = \langle M, M^\top \rangle_{L^2(\rho)}$, e.g., from a Cholesky decomposition.

2. Find a $\hat{\Gamma} = D^\top \Gamma$ with $\hat{\Gamma}^\top \hat{\Gamma} = I_n$, such that
   \[
   \hat{\Gamma}^\top D^\top \Sigma D \hat{\Gamma} \quad (= \hat{\Gamma}^{-1} D^\top \Sigma D \hat{\Gamma}^{-\top} = \Lambda^M)
   \]
   be diagonal (eigenvalue decomposition).

3. The diagonal of (8) contains the sought eigenvalues, and the principal component polynomials result by transforming the monomials using $\Gamma = D^{-\top} \hat{\Gamma}$.
Example of principal components

Figure: Example of the first three principal components of zero rates, representing the typical yield curve movements shift, steepening and curvature.
Pseudo principal components: motivation

Pseudo principal components are polynomials (linear combinations of monomials $M_i$) with could be principal components, but are actually invented.

Why pseudo?
- The model specification (1) is transparent, but at the same time exhibits unintuitive weights $u$, e.g. $u = (-1, 1.5, -2, 2.5, -3)^\top$
- Need to compute sensitivities wrt observables of the yield curve (e.g., modified duration as relative sensitivity to a parallel shift)

Approach 1: Linearly transform such that the yield curve can be specified in terms of zero rates at $n$ different nodes. Disadvantage: Changing one zero rate affects all other zero rates too (except at the nodes)

Solution: Use pseudo principal components!
Pseudo principal components: construction

The following construction principle can be used:

1. Construct the first (forward rate) pseudo PC $P_1 \in L^2(\rho)$ as the linear combination of $M_1, \ldots, M_n$ closest (in the norm of $L^2(\rho)$) to a constant function, normalize afterwards such that $\langle P_1, P_1 \rangle = 1$.

2. Construct the $l$-th pseudo PC orthonormal to the existing ones as linear combination of $M_1, \ldots, M_l$, e.g., $P_2(\tau) = p_1^{(2)} M_1(\tau) + p_2^{(2)} M_2(\tau)$ and $\langle P_1, P_2 \rangle = 0$, $\langle P_2, P_2 \rangle = 1$.

3. Any need to further adjust the pseudo PCs? Use orthonormal rotation matrices to rotate the basis set.
Figure: Example of the first three pseudo principal components of forward rates, representing the typical yield curve movements shift, steepening and curvature.
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Pricing zero bonds

Given the HJM parameterization of the forward rate curve, the price at time $t$ of a zero bond with maturity $T$ is

$$P(t, T) = \exp \left( -\tilde{A}(T - t) - \tilde{M}^\top (T - t)u(t) \right),$$

(9)

for $\tilde{A}(\tau) = \int_0^\tau A(s)ds$ and

$$\tilde{M}_i(\tau) = \int_0^\tau M(s)ds = \frac{1}{\alpha_i} (1 - \exp(-\alpha_i \tau)).$$
Pricing zero bond options (and thus caps/floors)

Recall $\Sigma_{t,T} \in \mathbb{R}^{n \times n}$ from (4) with

$$(\Sigma_{t,T})_{jk} = -\frac{\exp(-(\alpha_j + \alpha_k)(T - t)) - 1}{\alpha_j + \alpha_k} (\sigma\sigma^\top)_{jk}.$$ 

Theorem (Price of a European call option on a zero bond $P(t, S)$)

The price of the call at time $t$ with expiry $T < S$ and strike $K$ is

$$ZBC(t, T, S, K) = P(t, S)\Phi \left( \kappa + \frac{1}{2} \frac{C^2_{t, T, S}}{C_{t, T, S}} \right) - P(t, T)K\Phi \left( \kappa - \frac{1}{2} \frac{C^2_{t, T, S}}{C_{t, T, S}} \right),$$

where $\kappa = \ln \frac{P(t, S)}{P(t, T)K}$ and $C^2_{t, T, S} = \tilde{M}^\top (S - T)\Sigma_{t, T} \tilde{M}(S - T)$.

Proof: Using SDE of quotient process $d\frac{P(t, V)}{P(t, T)}$, and distribution of $u(T)$. 
Pricing zero bond options: $G2++$ formula

For the two-factor model, [Brigo and Mercurio(2006)] compute explicitly

$$C_{t,T,S} = \frac{\sigma^2}{2\alpha_1^3} \left[ 1 - e^{-\alpha_1(S-T)} \right]^2 \left[ 1 - e^{-2\alpha_1(T-t)} \right]$$

$$+ \frac{\eta^2}{2\alpha_2^3} \left[ 1 - e^{-\alpha_2(S-T)} \right]^2 \left[ 1 - e^{-2\alpha_2(T-t)} \right]$$

$$+ 2\rho \frac{\sigma\eta}{\alpha_1\alpha_2(\alpha_1 + \alpha_2)} \left[ 1 - e^{-\alpha_1(S-T)} \right] \left[ 1 - e^{-\alpha_2(S-T)} \right] \cdot$$

$$\left[ 1 - e^{-(\alpha_1+\alpha_2)(T-t)} \right]$$

($\sigma$, $\rho$, $\eta$ have different meanings there)

Imagine the size of the formula for $n = 5$ parameters...
Conclusion

- Tractable, transparent dynamic term-structure model as sum of exponentials with arbitrary degrees of freedom ideal for risk management
- Identified principal components in parametric affine models using weighted function spaces
- Pseudo principal components
  - ...add intuitive meaning to the weights / factor loadings, and
  - ...allow for sensitivities extending well-known ones (e.g. duration); these model-based sensitivities can be useful for hedging and portfolio management.
- Pricing caps and floors with transparent matrix-based formulas
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Overview

5 SDE solution of $u$

6 Calibration

7 Volatility and correlation
For $S = \text{diag}(-\alpha)$, the stochastic dynamics of the weights in (2), started in $t_0$ with $u(t_0)$, specifies an Ornstein-Uhlenbeck process with the solution

$$u(T) = \exp(S(T - t_0))u(t_0) + S^{-1}(\exp(S(T - t_0)) - 1)R$$
$$+ \int_{t_0}^{T} \exp(S(T - \nu))\sigma \, dW(\nu),$$

(10)

for any time $T \geq t_0$. 
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Calibration: How to get the numbers

Different parameters to be chosen / estimated / calibrated:

- Model specification: $n, \alpha \in \mathbb{R}^n$
- Static curve specification: $u \in \mathbb{R}^n$
- Specification of dynamics: $R \in \mathbb{R}^n, \sigma \in \mathbb{R}^{n \times n}$

There are two alternatives: implied calibration to market data and historical calibration, each one with its (dis)advantages.
Calibration: How to get the numbers (II)

Implied calibration to market data (e.g., for G2++):

1. Determine and interpolate yield curve $\sim$ all zero bond prices
2. Fix $n$, set $u = 0$, drop $R$. Adding time-dependent parameters ensures perfect fit of yield curve for any $\alpha$, $u$, $\sigma$
3. Calibrate to cap volatility surface $\sim \alpha$, $\sigma$

Historical calibration (no perfect fit of yield curve!):

1. Fix $n$, $\alpha$
2. Determine best fit of yield curve using (1) by minimizing an error function (e.g., RMSE) $\sim u^{(i)}$ for $i = 1, \ldots, N$
3. Estimate $R$, $\sigma$ from curve time series $u^{(i)} \sim R$, $\sigma$, affine term $A(\tau)$

Problem: Interpolation has to be done beforehand; today’s model inconsistent with yesterday’s model!

Problem: Historical calibration needs $A(\tau)$ in step 2, which depends on $\sigma$. $\Rightarrow$ Use iterative procedure!
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**Volatility and correlation**

*Figure*: Estimated volatility and correlation of annualized zero rates, based on German government bonds in the time period from Jan. 2002 to Dec. 2011.