Impulse Control for Jump-Diffusions: Viscosity Solutions of Quasi-Variational Inequalities and Applications in Bank Risk Management

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von Diplom-Finanz- und Wirtschaftsmathematiker Roland C. Seydel
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Abstract

This work is about combined stochastic and impulse control for jump-diffusions, its relation to Hamilton-Jacobi-Bellman quasi-variational inequalities (HJBQVIs), their numerical solution, and applications in bank risk management. An HJBQVI is a nonlinear partial integro-differential equation (PIDE) consisting of an HJB part (for stochastic control) combined with a nonlocal impulse intervention term; the jump measure in the PIDE may be singular, corresponding to a jump process of infinite activity.

General theorems for existence and uniqueness of viscosity solutions of such HJBQVIs are established in Chapter 2. We prove via stochastic means that the value function of stochastic and impulse control is an HJBQVI viscosity solution, whereas our uniqueness (comparison) results adapt techniques from viscosity solution theory. To our knowledge, our results are the first rigorous treatment of impulse control for jump-diffusion processes in a general viscosity solution framework. In the proofs, no prior continuity of the value function is assumed, quadratic costs are allowed, and elliptic and parabolic results are presented for solutions possibly unbounded at infinity.

In Chapter 3, we analyze the numerical solution of an HJBQVI via iterated optimal stopping. First, the viscosity characteristics of the value function of combined stochastic control and stopping is established by similar techniques as in Chapter 2. We prove that the value functions of iterated optimal stopping as solutions of suitable HJB variational inequalities converge to the unique HJBQVI viscosity solution.

As an application of stochastic and impulse control, in Chapter 4 we propose a dynamic model where a bank controls its leverage by securitization of loans. The fixed transaction costs in such a securitization lead to a formulation in an impulse control framework. The bank operates in a Markov-switching economy and wants to maximize the utility of terminal equity value. For the quasi-variational inequality associated with this impulse control problem, we prove existence and uniqueness of viscosity solutions. Iterated optimal stopping is used to find a numerical solution of this PDE, and numerical examples are discussed.

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Contents

Introduction

There are three basic settings how to control dynamically a stochastic process: Stochastic control, singular control, and impulse control. These three concepts in finance usually correspond to no transactions, only proportional transaction costs, and fixed transaction costs, respectively.

Impulse control in a way can be seen as the most general of the three notions, as it also includes variable transaction costs. Formally, impulse control can be reduced to singular control by letting the fixed costs converge to zero (e.g., Øksendal [90]), and singular control arises as natural extension of some stochastic control problems for unbounded control sets (e.g., Pham [97]).

Many decisions in real life can only be made if one tacitly accepts external or internal intervention costs. This is particularly true for financial markets, where transaction costs can be usually assigned a specific monetary value. Besides, the discrete interventions arising from fixed transaction costs are natural because they correspond to the decisions of humans or organizations which cannot act continuously, even not approximately. But even if there are transaction costs, this does not mean that stochastic control is superfluous as a model. The crucial question in modelling is whether transaction costs can be neglected or not, and whether acting continuously is a reasonable approximation to reality.

This thesis is about combined stochastic and impulse control and its link to certain partial integro-differential equations. Using combined stochastic and impulse control as model is reasonable if there are two means of controlling the system, one where transaction costs can be neglected, and another where fixed costs are an important ingredient.

In finance, transaction costs lead to incomplete markets. So incomplete markets are a natural area of application for impulse control, but of course applications are not limited to financial mathematics. An introduction to impulse (and stochastic) control with numerous examples and applications can be found in Bensoussan and Lions [17], Øksendal and Sulem [93]; see also Fleming and Soner [46]. Further applications appearing in the literature are for instance option pricing with transaction costs ([110], [36], [21]), optimal portfolios ([71], [100], [95], [92]), options in long-term insurance contracts [29], or combined control of an exchange rate by the Central Bank ([87], [23]); for some complements see the overview in Korn [72].

Although we are mainly concerned with applications in finance, the mathematical part of our thesis is generic and can be applied to problems from any field.

This work consists of three main parts, all centering on stochastic impulse control of jump-diffusion processes. The setting of a combined stochastic and impulse control problem is introduced in Chapter 2, and it is proved that the value function of such a problem is the unique viscosity solution of a Hamilton-Jacobi-Bellman quasi-variational inequality (HJBQVI). In Chapter 3, we investigate numerical methods for solving this nonlocal partial integro-differential equation, concentrating on a method called iterated optimal stopping. Finally, in the last main chapter, we present a model of a bank utilizing combined impulse and stochastic control to dynamically control the leverage of its lending activities.

Our thesis is complemented by a short introductory chapter on viscosity solution theory, and a summary and conclusion at the end.

Impulse control of jump-diffusions and viscosity solutions of HJBQVIs

The aim in combined stochastic and impulse control is to maximize a certain functional, dependent on a controlled stochastic process X^α until some exit time τ:

$$v(t,x) := \sup_\alpha \mathbb{E}^{(t,x)} \left[\int_t^\tau f(s, X_s^\alpha, \beta_s) ds + g(\tau, X_\tau^\alpha) 1_{\tau<\infty} + \sum_{\tau_j \leq \tau} K(\tau_j, X_{\tau_j-}^\alpha, \zeta_j) \right] \tag{1}$$

Here $\alpha = (\beta, \gamma)$, with β the stochastic control process, and $\gamma = (\tau_i, \zeta_i)_{i\geq 1}$ the impulse control strategy. The function K is typically negative and incorporates the impulse transaction costs, and the functions f and g are profit functions. In our case, X^α follows a jump-diffusion stochastic differential equation (SDE) whose dynamics is continuously controlled by β, and which is manipulated at discrete impulse intervention times τ_i to go from X_{τ_i-} to a new state $\Gamma(t, X_{\tau_i-}, \zeta_i)$. The SDE dynamics between the impulses is given by

$$dX_t = \mu(t, X_{t-}, \beta_{t-})\, dt + \sigma(t, X_{t-}, \beta_{t-})\, dW_t + \int \ell(t, X_{t-}, \beta_{t-}, z)\, \overline{N}(dz, dt), \tag{2}$$

where W is a standard Brownian motion, and $\overline{N}(dz,dt) = N(dz,dt) - 1_{|z|<1}\nu(dz)dt$ is a compensated Poisson random measure with possibly unbounded intensity measure ν. Examples for X are standard Brownian motion, a Lévy process, or a point process with state-dependent finite jump intensity.

The main purpose of Chapter 2 is to prove that the value function v of (1) for $\tau = T \wedge \tau_S$ is the unique viscosity solution of the following partial integro-differential equation (PIDE), a so-called Hamilton-Jacobi-Bellman quasi-variational inequality (HJBQVI):

$$\min(-\sup_{\beta\in B}\{u_t + \mathcal{L}^\beta u + f^\beta\}, u - \mathcal{M}u) = 0 \qquad \text{in } [0,T)\times S, \tag{3}$$

together with suitable boundary conditions. Here, $\mathcal{L}^\beta$ is the infinitesimal generator of the controlled SDE,

$$\begin{aligned}\mathcal{L}^\beta u(t,x) = \frac{1}{2} tr\left(\sigma(t,x,\beta)\sigma^T(t,x,\beta) D_x^2 u(t,x)\right) + \langle \mu(t,x,\beta), \nabla_x u(t,x)\rangle \\ + \int u(t, x + \ell(t,x,\beta,z)) - u(t,x) - \langle \nabla_x u(t,x), \ell(t,x,\beta,z)\rangle 1_{|z|<1}\, \nu(dz),\end{aligned}$$

and $\mathcal{M}$ is the intervention operator selecting the momentarily best impulse,

$$\mathcal{M}u(t,x) = \sup_\zeta \{u(t, \Gamma(t,x,\zeta)) + K(t,x,\zeta)\}.$$

If v is sufficiently regular, then this viscosity solution is also a classical solution. Because this regularity of v is not known or even true in general, a weak solution concept has to be employed.

It turns out that the right concept is precisely that of viscosity solutions (see Crandall and Lions [31], Crandall et al. [33]; an introduction is given in Chapter 1).

Equation (3) in our case is formally a nonlinear, nonlocal, possibly degenerate, second order parabolic PIDE. We point out that the investigated stochastic process is allowed to have jumps, including so-called "infinite-activity processes" where the jump measure ν may be singular at the origin. (It can be argued that infinite-activity processes are a good model for stock prices, see, e.g., Cont and Tankov [30], Eberlein and Keller [41].)

The existence and uniqueness result of Chapter 2 is well known in the diffusion case, and also in some special cases for jump processes with finite activity. Our contribution is to prove this result in a very general setting for jump-diffusion processes, including a nonlocal boundary with feedback terms. While the existence for (3) (and its elliptic counterpart) is proved by stochastic means, the uniqueness part uses results from viscosity solution theory. There are two main technical difficulties to overcome: (a) The process may leave S by a jump and be forced to return by an impulse; (b) ν may have a singularity at 0. While (a) leads us to impose special nonlocal feedback boundary conditions for (3), (b) implies that we cannot use the standard *maximum principle for semicontinuous functions* in the uniqueness proof, but rather we have to use a nonlocal version of it, recently proved by Barles and Imbert [11].

Main assumptions for our result are (local Lipschitz) continuity of the functions involved, continuity of v at the boundary, and existence of a strict viscosity supersolution of (3). In the proofs, quadratic costs are allowed, and elliptic and parabolic results are presented for solutions possibly unbounded at infinity; the continuity of the value function follows from the uniqueness result.

The contents of Chapter 2 correspond in large parts to the working paper Seydel [106], a shortened version of which was accepted for publication in "Stochastic Processes and their Applications" [107].

Numerical schemes for (HJB)QVIs and iterated optimal stopping

While simple one-dimensional elliptic quasi-variational inequalities can typically be solved by hand by applying the smooth-fit property (see, e.g., Øksendal and Sulem [93]), the smooth-fit technique in general fails for parabolic QVIs, or elliptic QVIs in higher dimensions. This is frequently not because of missing C^1 regularity (see Bensoussan and Lions [17], §4.2 and Guo and Wu [57], Pham [98] for regularity results for QVIs), but rather because $\mathbb{R}^d$ is not ordered for $d \geq 2$.

This shows clearly the need for numerical methods for QVIs. In Chapter 3, we first survey a few selected methods, then concentrating on a technique called *iterated optimal stopping*. Starting with an iterate v^0 (the value function without impulses), the value function v^n of every subsequent iteration step $n \geq 1$ includes one more optimally placed impulse via the stopping payoff $\mathcal{M}v^{n-1}$. The combined stochastic control and stopping problem for $n \geq 1$ in the parabolic case corresponds to the HJB variational inequality (HJBVI)

$$\min(-\sup_{\beta \in B}\{u_t + \mathcal{L}^\beta u + f^\beta\}, u - \mathcal{M}v^{n-1}) = 0 \qquad \text{in } [0,T) \times S, \tag{4}$$

together with suitable boundary conditions.

We establish in Chapter 3 first the relation between combined stochastic control and stopping problems and HJBVIs such as (4) by proving viscosity solution existence and uniqueness, similarly to Chapter 2. Our result is new in this general setting, although there are previous results by Pham [99] for stochastic control and stopping of jump-diffusions in the case $S = \mathbb{R}^d$.

Combining stability of viscosity solutions of (4) with stochastic techniques, we prove the convergence of the optimal stopping iterate v^n to the value function of combined stochastic and impulse control. This is the first such convergence result for jump-diffusions using viscosity solution techniques; the assumptions are mainly those of the first part. As byproduct of our convergence proof, we demonstrate an interesting new equivalence of two optimal stopping iterations for problems with exit time, using their HJBVI viscosity characteristics.

Our chapter about numerical schemes ends with the description of an example implementation of iterated optimal stopping and a convergence analysis of the discrete approximation.

A dynamic model of credit securitization

As an application of combined stochastic and impulse control, we consider in Chapter 4 a new model for the dynamic risk management decisision of a commercial bank. This bank issues loans to customers with money borrowed partly on the capital market; the loans may not be paid back (default), but on the other hand offer an expected excess return over the risk-free interest rate. To manage the risks optimally, the bank can issue new loans (via stochastic control), or can reduce its exposure by selling loans (securitization impulse).

The term *securitization* describes the process of packaging loans into financial instruments, such as Asset-Backed Securities (ABS). Those financial instruments became notorious when the poor quality of some of the packaged loans triggered the financial crisis of 2008. A securitization always comes with certain fixed costs (rating agency fees, legal costs, etc.), and thus investigating the problem in an impulse control framework is natural. Moreover, there may be variable transaction costs reflecting that the market price of the loans to be sold may change with an economic factor process.

We investigate the model from a mathematical and economic point of view. Mathematically, our impulse control problem is a time-dependent problem on a three-dimensional stochastic process with jumps (loan exposure, cash, economic factor), and its value function can be shown to be the unique continuous viscosity solution of an HJBQVI, using the results from Chapter 2. A numerical solution by iterated optimal stopping is carried out, and the results are interpreted from an economic perspective.

The main economic conclusion is that fixed and variable transaction costs strongly influence the securitization decision, and a bank may have an incentive to keep its loans during a recession although they generate losses.

The contents of Chapter 4 correspond in large parts to the working paper Frey and Seydel [50], which will be submitted to “Mathematics and Financial Economics”.

Notation

Let us introduce some notation used throughout the thesis. The term *elliptic* is here understood in the wide sense of a weakly elliptic (integro-)differential operator without time-dependence; *parabolic* means weakly parabolic with time-dependence. Sometimes the abbreviation *wlog* is used in proofs and means *without loss of generality.*

The natural and integer numbers are denoted by $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, and $\mathbb{Z} = \mathbb{N}_0 \cup (-\mathbb{N})$. $\mathbb{R}^d$ for $d \geq 1$ is the Euclidean space equipped with the usual norm and the scalar product denoted by $\langle a, b\rangle$ or $a \cdot b$ for $a, b \in \mathbb{R}^d$. The open ball around $x \in \mathbb{R}^d$ with radius ε is denoted by $B(x, \varepsilon)$. For sets $A, B \subset \mathbb{R}^d$, the notation $A \subset\subset B$ (compactly embedded) means that $\overline{A} \subset B$, and $A^c = \mathbb{R}^d \setminus A$ is the complement of A. We denote the space of symmetric matrices $\subset \mathbb{R}^{d\times d}$ by $\mathbb{S}^d$, $\geq$ is the usual partial ordering in $\mathbb{R}^{d\times d}$, i.e. $X \geq Y \Leftrightarrow X - Y$ positive semidefinite. $|\cdot|$ on $\mathbb{S}^d$ is the usual eigenvalue norm. $C^2(\mathbb{R}^d)$ is the space of all functions two times continuously differentiable with values in $\mathbb{R}$, and as usual, u_t or $\dot{u}$ denotes the time derivative of u. $L^2(\mathbb{P}; \mathbb{R}^d)$ is the Hilbert space of all $\mathbb{P}$-square-integrable measurable random variables with values in $\mathbb{R}^d$, the measure $\mathbb{P}^X = \mathbb{P} \circ X^{-1}$ is sometimes used to lighten notation. The positive and negative parts of a function f are denoted by $f^+ = \max(f, 0)$ and $f^- = -\min(f, 0) = -(f \wedge 0)$, respectively, such that $f = f^+ - f^-$. If A is a matrix, then A_{i*} denotes the i-th row of this matrix.

For the convenience of the reader, we include also an index of notation introduced in the course of this work:

Chapter 1

Viscosity solutions of PDEs

This short introductory chapter sketches the main notions and tools of viscosity solution theory, as employed in this thesis. Although we focus on partial differential equations (PDEs), we highlight differences to the case of PIDEs including integral terms. We first introduce continuous viscosity solutions, collect some properties of semicontinuous functions, and then discuss briefly the concept of discontinuous viscosity solutions.

In the presentation of this chapter, we follow the "User's guide" [33] and the lecture notes Barles [10], which the reader may consult for further details. Other references in the context of HJB equations of first or second order are Barles [9], Bardi and Capuzzo-Dolcetta [8] and Fleming and Soner [46].

1.1 Continuous viscosity solutions

The concept of viscosity solutions originated in the analysis of first-order Hamilton-Jacobi equations, when a small perturbation term $-\varepsilon\Delta u$ was added to the first-order PDE to establish an existence and uniqueness result for $\varepsilon \to 0$, a method known as "vanishing viscosity"; see Crandall and Lions [31], Lions [78], or Crandall et al. [32]. The basic idea of viscosity solutions is to replace the derivatives in a PDE pointwise by the derivatives of a suitable smooth function φ, which is related to v by a maximum of $v - \varphi$ (viscosity subsolution) or minimum (viscosity supersolution).

Why is one interested in viscosity solutions? The first argument is as for any weak solution concept: By weakening the regularity assumptions, we want to be able to apply existence and uniqueness results to a broader class of equations; in a second step, regularity may still be shown. Viscosity solutions have proved to be very successful for many problems not only of Hamilton-Jacobi type; important in our context is that viscosity solution theory encompasses weakly elliptic or parabolic equations, which is not the case for weak variational solutions in Sobolev spaces. Compared to this latter solution concept, a further appeal of viscosity solutions is that they are defined pointwise and can thus be understood without much background in functional analysis.[1]

Nonetheless, viscosity solution theory comes with its own toolbox of theorems and other devices, which are sometimes difficult to understand. In the following, we will introduce a few essential

[1] For relations between these two solution concepts, see Ishii [64].

notions and tools in a PDE context, pointing out how they have to be adapted to include nonlocal integral terms.

Consider a general second-order PDE of the form

$$F(x, u(x), Du(x), D^2u(x)) = 0, \qquad x \in \mathscr{O} \tag{1.1}$$

where $F : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d \to \mathbb{R}$ is continuous, $\mathbb{S}^d$ the set of symmetric $d \times d$ matrices, and $\mathscr{O} \subset \mathbb{R}^d$ is an arbitrary set.

Definition 1.1.1. *A function $u \in C(\mathscr{O})$ is a* viscosity subsolution *of* (1.1) *if for any point $x_0 \in \mathscr{O}$ and all $\varphi \in C^2(\mathscr{O})$ such that $u - \varphi$ has a local maximum in x_0 relative to $\mathscr{O}$,*

$$F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \leq 0.$$

The function u is a viscosity supersolution *of* (1.1) *if for any point $x_0 \in \mathscr{O}$ and all $\varphi \in C^2(\mathscr{O})$ such that $u - \varphi$ has a local minimum in x_0 relative to $\mathscr{O}$,*

$$F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \geq 0.$$

Finally, u is called a viscosity solution *if it is a sub and supersolution. (We will sometimes omit the part* viscosity *if there is no risk of confusion.)*

By a simple application of the first and second order conditions of maximum and minimum, any classical solution $u \in C^2(\mathbb{R}^d)$ of (1.1) is also a viscosity solution, provided that the following degenerate ellipticity condition holds:

$$F(x, r, p, X) \leq F(x, r, p, Y) \qquad \text{whenever } Y \leq X. \tag{1.2}$$

Here $Y \leq X$ denotes the usual partial ordering in $\mathbb{S}^d$. Another condition which is generally imposed is

$$F(x, r, p, X) \leq F(x, s, p, X) \qquad \text{whenever } r \leq s. \tag{1.3}$$

The conditions (1.2) and (1.3) make sure that for typical equations we cannot modify the viscosity solution definition arbitrarily by multiplying F with -1. If (1.2) and (1.3) hold, then F is termed *proper*.

There is another definition of viscosity solutions, which is equivalent to Def. 1.1.1 in a PDE context, using an appropriate replacement for the "derivative" of a nondifferentiable function $u : \mathbb{R}^d \to \mathbb{R}$. It is based on the semijets

$$\begin{aligned}
J^+_{\mathscr{O}}u(x) = \{(p, X) \in \mathbb{R}^d \times \mathbb{S}^d : u(x+z) \leq u(x) + \langle p, z\rangle + \frac{1}{2}\langle Xz, z\rangle + o(|z|^2) \\
\text{as } z \to 0, x + z \in \mathscr{O}\}, \\
J^-_{\mathscr{O}}u(x) = \{(p, X) \in \mathbb{R}^d \times \mathbb{S}^d : u(x+z) \geq u(x) + \langle p, z\rangle + \frac{1}{2}\langle Xz, z\rangle + o(|z|^2) \\
\text{as } z \to 0, x + z \in \mathscr{O}\}.
\end{aligned}$$

If the set $\mathscr{O}$ in question is clear, or $x \in int(\mathscr{O})$, then we will omit the subscript $\mathscr{O}$. If u is twice differentiable at $x \in int(\mathscr{O})$, then $J^+u(x) \cap J^-u(x) = \{(Du(x), D^2u(x))\}$. The relation to the previous notion is for $x_0 \in \mathscr{O}$

$$J^+_{\mathscr{O}}u(x_0) = \{(D\varphi(x_0), D^2\varphi(x_0)) : \varphi \in C^2,\ u - \varphi \text{ has local maximum at } x_0\},$$

and similarly for $J^-_{\mathscr{O}}u(x_0)$, using $J^-_{\mathscr{O}}u(x) = -J^+_{\mathscr{O}}(-u)(x)$. A function $u \in C(\mathscr{O})$ is a viscosity subsolution of (1.1) iff

$$F(x_0, u(x_0), p, X) \le 0 \qquad \text{for all} \quad x_0 \in \mathscr{O}, (p, X) \in J^+_{\mathscr{O}}u(x_0);$$

it is a viscosity supersolution of (1.1) iff

$$F(x_0, u(x_0), q, Y) \ge 0 \qquad \text{for all} \quad x_0 \in \mathscr{O}, (q, Y) \in J^-_{\mathscr{O}}u(x_0).$$

Some "without loss of generalities". In Def. 1.1.1, one can without loss of generality replace $\varphi \in C^2$ by $\varphi \in C^k$ for any $k \ge 2$. Furthermore, "local maximum" may be replaced by "strict local maximum", by "global maximum" or by "strict global maximum", and similarly for the minimum. All these replacements are consequence of the local definition of the PDE: By adding to φ the function $\chi(x) = |x - x_0|^4$, which satisfies $\chi(x_0) = 0$, $D\chi(x_0) = 0$ and $D^2\chi(x_0) = 0$, the local maximum in x_0 turns into a strict local maximum. For the last replacement, if $u - \varphi$ has a local maximum in x_0, then because of the local boundedness of $u \in C(\mathscr{O})$ we can find a function $\tilde{\varphi}$ with $\tilde{\varphi} = \varphi$ locally around x_0 such that $u - \tilde{\varphi}$ has a global maximum in x_0, and $F(x_0, u(x_0), D\tilde{\varphi}(x_0), D^2\tilde{\varphi}(x_0)) = F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0))$.

Another equivalent formulation is to replace "global maximum of $u - \varphi$" by $u \le \varphi$, $u(x_0) - \varphi(x_0) = 0$, as adding constants to φ does not change the value of its derivatives.

All the above equivalent reformulations are not obvious anymore as soon as we add nonlocal terms to the PDE, e.g., if we investigate a partial integro-differential equation (PIDE) of the type

$$F(x, u(x), Du(x), D^2u(x), u(\cdot)) = \rho u(x) - D^2u(x) - \int u(x+z) - u(x)\, \nu(dz), \tag{1.4}$$

where the notation $u(\cdot)$ emphasizes that also nonlocal values of u are taken into account. If we can consider the entire integral with a continuous function u as argument, then again the function $\varphi \in C^2(\mathbb{R}^d)$ enters the equation only locally with $D\varphi(x_0)$ and $D^2\varphi(x_0)$; if this is not possible (e.g., if the integral requires smoothness of the integrand), then the discussion becomes a bit more complicated. We are in the latter case in §2.5.

Link to stochastic processes. The definition of a viscosity solution is intimately connected with stochastic processes. Consider the linear PIDE

$$-u_t - \mathcal{L}u = 0 \qquad \text{in } \mathbb{R}^+ \times \mathbb{R}^d, \tag{1.5}$$

and let X be a Markov process with infinitesimal generator $\mathcal{L}$. Then there is a correspondence

$$v \text{ is viscosity subsolution of (1.5)} \leftrightarrow v(t, X_t) \text{ is (local) submartingale},$$

and a correspondence between supersolution and supermartingale. For $v \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^d)$ with bounded derivatives, the submartingale property can be derived from the subsolution property by Dynkin's formula:

$$\mathbb{E}^{(t_0,x_0)}[v(t, X_t)] = v(t_0, x_0) + \mathbb{E}^{(t_0,x_0)}\left[\int_{t_0}^t (v_t + \mathcal{L}v)(s, X_s)ds\right] \ge v(t_0, x_0).$$

The inverse direction for general v and for stochastic and impulse control can be inferred from the theorems of Chapter 2.

A key advantage of viscosity solutions for optimal control problems is that under mild conditions, the value function is already a viscosity solution (necessary condition). This reverses the traditional way of thinking in verification theorems, where a sufficiently smooth solution can in theory be proved to be the seeked value function (sufficient criterion). However, the viscosity solution approach is feasible only if the uniqueness of the viscosity solution can be demonstrated — only then can we be sure that the viscosity solution actually is equal to the value function.

1.2 Tools for uniqueness proofs

In this section, we present a few standard devices that are used in typical uniqueness proofs for PDEs. We confine our discussion to the uniqueness case because viscosity solution existence in our work is proved by stochastic means.

We denote by $USC(\mathscr{O})$ the set of upper semicontinuous (usc) functions $u : \mathscr{O} \to \mathbb{R}$, i.e., functions with $\limsup_{k\to\infty} u(x_k) \le u(x_0)$ for any sequence $\mathscr{O} \ni x_k \to x_0 \in \mathscr{O}$; in the same way, $LSC(\mathscr{O})$ is the set of lower semicontinuous (lsc) functions on $\mathscr{O}$, i.e., $\liminf_{k\to\infty} u(x_k) \ge u(x_0)$ for any sequence $\mathscr{O} \ni x_k \to x_0 \in \mathscr{O}$.

The limiting semijets are defined by

$$\begin{aligned}\overline{J}^{+}_{\mathscr{O}} u(x) = \{(p,X) \in \mathbb{R}^d \times \mathbb{S}^d :\ & \text{there exist } (\mathscr{O},\mathbb{R}^d,\mathbb{S}^d) \ni (x_k,p_k,X_k) \to (x,p,X), \\ & (p_k,X_k) \in J^{+}_{\mathscr{O}} u(x_k) \text{ such that } u(x_k) \to u(x)\},\end{aligned}$$

and similarly for $\overline{J}^{-}_{\mathscr{O}} u(x)$.

The analytical heart of the theory is the following maximum principle for semicontinuous functions, which is a generalization of a classical maximum principle:

Theorem 1.2.1 (Theorem 3.2 in Crandall et al. [33]). *Let $\mathscr{O}_i$ be a locally compact subset of $\mathbb{R}^{d_i}$ for $i = 1, \dots, k$, $\mathscr{O} = \mathscr{O}_1 \times \dots \times \mathscr{O}_k$, $u_i \in USC(\mathscr{O}_i)$, and φ be twice continuously differentiable in a neighbourhood of $\mathscr{O}$. Set*

$$w(x) = u_1(x_1) + \dots + u_k(x_k) \quad \text{for } x = (x_1, \dots, x_k) \in \mathscr{O},$$

and suppose $\hat{x} = (\hat{x}_1, \dots, \hat{x}_k) \in \mathscr{O}$ is a local maximum of $w - \varphi$ relative to $\mathscr{O}$. Then for each $\varepsilon > 0$ there exists $X_i \in \mathbb{S}^{d_i}$ such that

$$(D_{x_i}\varphi(\hat{x}), X_i) \in \overline{J}^{+}_{\mathscr{O}_i} u_i(\hat{x}_i) \quad \text{for } i = 1, \dots, k,$$

and the block diagonal matrix with entries X_i satisfies

$$-\left(\frac{1}{\varepsilon} + \|A\|\right) I \le \begin{pmatrix} X_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & X_k \end{pmatrix} \le A + \varepsilon A^2, \tag{1.6}$$

where $A = D^2\varphi(\hat{x}) \in \mathbb{S}^d$, $d = d_1 + \dots + d_k$.

Define the function $\hat{\varphi}_i : \mathbb{R}^{d_i} \to \mathbb{R}$ by $\hat{\varphi}_i(y) = \varphi(y, \hat{x}_{-i})$, where $(\hat{x}_i, \hat{x}_{-i}) = \hat{x}$. Then under the conditions of the theorem, $u_i - \hat{\varphi}_i$ has a local maximum in $\hat{x}_i$. Thus using the characterization of J^+u, we can immediately deduce

$$(D_{x_i}\varphi(\hat{x}), D_{x_i x_i}\varphi(\hat{x})) \in J^+_{\mathscr{O}_i} u_i(\hat{x}_i).$$

This shows that the real power of Theorem 1.2.1 lies in the matrix inequality (1.6).

For a PIDE with singular integral terms, the above maximum principle has to be replaced by a different, nonlocal one including additional smooth functions φ_k; see Barles and Imbert [11] or §2.5.

We continue with a lemma which is useful for uniqueness proofs:

Lemma 1.2.2 (Lemma 3.1 in Crandall et al. [33]). *Let $\mathscr{O}$ be a subset of $\mathbb{R}^d$, $u \in USC(\mathscr{O})$, $v \in LSC(\mathscr{O})$ and*

$$M_\alpha = \sup_{\mathscr{O}\times\mathscr{O}} \{u(x) - v(y) - \frac{\alpha}{2}|x-y|^2\}$$

for $\alpha > 0$. Let $M_\alpha < \infty$ for large α and (x_α, y_α) be such that

$$\lim_{\alpha\to\infty} \{M_\alpha - u(x_\alpha) - v(y_\alpha) - \frac{\alpha}{2}|x_\alpha - y_\alpha|^2\} = 0.$$

Then $\lim_{\alpha\to\infty} \alpha|x_\alpha - y_\alpha|^2 = 0$, and

$$\lim_{\alpha\to\infty} M_\alpha = u(\hat{x}) - v(\hat{x}) = \sup_{\mathscr{O}}\{u(x) - v(x)\} \text{ for any limit point } \hat{x} \in \mathscr{O} \text{ of } (x_\alpha).$$

Let $\Omega \subset \mathbb{R}^d$ be an open set. To prove uniqueness of viscosity solutions, one usually proves a comparison theorem, which states for a usc subsolution u and an lsc supersolution v that

$$u \leq v \qquad \text{in } \overline{\Omega},$$

provided that $u \leq v$ holds on the boundary $\partial\Omega$. This comparison in general is true if (a) F is proper, and (b) there is a $\gamma > 0$ s.th.

$$\gamma(r-s) \leq F(x,r,p,X) - F(x,s,p,X) \quad \text{for } r \geq s, (x,p,X) \in \overline{\Omega} \times \mathbb{R}^d \times \mathbb{S}^d. \tag{1.7}$$

Furthermore, (c) there exist a modulus of continuity $\omega : [0,\infty] \to [0,\infty]$ that satisfies $\omega(0-) = 0$ such that

$$F(y, r, \alpha(x-y), Y) - F(x, r, \alpha(x-y), X) \leq \omega(\alpha|x-y|^2 + |x-y|) \tag{1.8}$$

whenever $x, y \in \Omega$, $r \in \mathbb{R}$, $X, Y \in \mathbb{S}^d$, and the matrix inequality

$$-3\alpha \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \tag{1.9}$$

holds.

Roughly, the proof of a comparison theorem works in the following way: We assume to prove by contradiction that $\sup_{x\in\Omega}\{u(x) - v(x)\} > 0$. As first step, we consider the function $u(x) - v(y) - \frac{\alpha}{2}|x-y|^2$ and prove that it assumes a maximum in a (x_α, y_α) lying in a compact set independent of α; then we apply Lemma 1.2.2 to conclude that $\frac{\alpha}{2}|x_\alpha - y_\alpha|^2 \to 0$ for $\alpha \to \infty$. Finally, we apply Theorem 1.2.1 to obtain the matrix inequality (1.9), and use (1.7), (1.8) and that F is proper to obtain the contradiction in a sequence of inequalities.

Some properties of semicontinuous functions. As there is no obvious reference for a collection of facts about semicontinuous functions, we provide some properties here, most of which are used in §2.5. Let $u : \mathbb{R}^d \to \mathbb{R}$.

(i) u usc $\Leftrightarrow -u$ lsc

(ii) u usc $\Leftrightarrow \{u \geq b\}$ closed for any $b \in \mathbb{R}$. In the same way, u lsc $\Leftrightarrow \{u \leq b\}$ closed for any $b \in \mathbb{R}$

(iii) A usc function assumes its maximum on each compact; a usc function assumes its minimum on each compact.

(iv) For a closed set C, the indicator function 1_C is usc; for an open set O, 1_O is lsc.

(v) If u, v usc and $u, v \geq 0$, then uv usc; if u, v lsc and $u, v \geq 0$, then uv lsc

1.3 Discontinuous viscosity solutions

So far, we have not seen any boundary conditions in the formulation of viscosity solutions. We will now introduce the concept of *discontinuous viscosity solutions*, which permits easy inclusion of general boundary conditions already in the problem formulation, and works with possibly discontinuous solutions u. Let $\Omega \subset \mathbb{R}^d$ for this section be an open set. Then one considers the equation

$$G(x, u(x), Du(x), D^2u(x)) = 0, \qquad x \in \overline{\Omega}, \tag{1.10}$$

where this time, $G : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d \to \mathbb{R}$ is locally bounded, but may be discontinuous. To give an important example, we can include Dirichlet boundary conditions g by setting

$$G(x, r, p, X) = \begin{cases} F(x, r, p, X) & x \in \Omega \\ r - g(x) & x \in \partial\Omega. \end{cases} \tag{1.11}$$

Let $\mathscr{O} \subset \mathbb{R}^d$ be an arbitrary set. For a locally bounded function $u : \mathscr{O} \to \mathbb{R}$, we define the upper and lower semicontinuous envelopes for $x \in \mathscr{O}$

$$u^*(x) = \limsup_{\substack{y \to x \\ y \in \mathscr{O}}} u(y), \qquad u_*(x) = \liminf_{\substack{y \to x \\ y \in \mathscr{O}}} u(y).$$

The usc envelope u^* is the smallest upper semicontinuous function $\geq u$, and u_* the largest lsc function $\leq u$. The envelopes applied to G are always taken with respect to *all* arguments.

Definition 1.3.1. *A locally bounded function $u : \overline{\Omega} \to \mathbb{R}$ is a* discontinuous viscosity subsolution *of* (1.10) *if for any point $x_0 \in \overline{\Omega}$ and all $\varphi \in C^2(\overline{\Omega})$ such that $u^* - \varphi$ has a local maximum in x_0,*

$$G_*(x_0, u^*(x_0), D\varphi(x_0), D^2\varphi(x_0)) \leq 0.$$

The function u is a discontinuous viscosity supersolution *of* (1.10) *if for any point $x_0 \in \overline{\Omega}$ and all $\varphi \in C^2(\overline{\Omega})$ such that $u_* - \varphi$ has a local minimum in x_0,*

$$G^*(x_0, u_*(x_0), D\varphi(x_0), D^2\varphi(x_0)) \geq 0.$$

Finally, u is called a discontinuous viscosity solution *if it is a discontinuous sub and supersolution.*

Note that in the usual setting of (1.11) with F and g continuous, the envelopes G^* and G_* are equal to G except on ∂S, where they are $\max(F(x,r,p,X), r-g(x))$ and $\min(F(x,r,p,X), r-g(x))$, respectively.

A comparison result for discontinuous viscosity solutions states for a subsolution u and a supersolution v

$$u^* \leq v_*,$$

in Ω or $\overline{\Omega}$. If such a comparison holds, we can immediately deduce that any discontinuous viscosity solution u is continuous, because $u_* \leq u^*$ holds by definition. Then why, one might ask, not simply investigate continuous viscosity solutions? One reason is that weakening assumptions makes it perhaps easier to demonstrate existence, as for any weak solution concept; in this respect, discontinuous viscosity solutions are just used as a tool to simplify proofs. Another reason is the increased flexibility with respect to boundary handling, which permits the investigation of solutions discontinuous at the boundary. And finally, a stability result can be stated elegantly for discontinuous viscosity solutions, as follows below.

But first, we need another limit definition. Let $(u_\varepsilon)_{\varepsilon>0}$ be a sequence of uniformly locally bounded functions $u_\varepsilon : \mathscr{O} \to \mathbb{R}$. Then $\overline{u}$, $\underline{u}$ are defined by

$$\overline{u}(x) = (\limsup)^* u_\varepsilon(x) = \limsup_{\substack{y\to x, y\in\mathscr{O}\\ \varepsilon\to 0}} u_\varepsilon(y), \qquad \underline{u}(x) = (\liminf)_* u_\varepsilon(x) = \liminf_{\substack{y\to x, y\in\mathscr{O}\\ \varepsilon\to 0}} u_\varepsilon(y).$$

By their definition, $\overline{u} \in USC(\mathscr{O})$, and $\underline{u} \in LSC(\mathscr{O})$.

Theorem 1.3.2 (Barles [9], Theorem 4.1)**.** *Assume that for $\varepsilon > 0$, $(u_\varepsilon)_\varepsilon$ is a sequence of discontinuous subsolutions (supersolutions) of*

$$G_\varepsilon(x, u, Du, D^2u) = 0 \qquad \text{on } \overline{\Omega}$$

which is uniformly locally bounded in $\overline{\Omega}$. We assume that $(G_\varepsilon)_\varepsilon$ are uniformly locally bounded in $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d$ and satisfy the ellipticity condition (1.2)*. Then $\overline{u}$ ($\underline{u}$) is a discontinuous subsolution (supersolution) of*

$$\underline{G}(x, u, Du, D^2u) = 0 \qquad \text{on } \overline{\Omega}$$
$$(\overline{G}(x, u, Du, D^2u) = 0 \qquad \text{on } \overline{\Omega}).$$

Note that in fact, no convergence of derivatives is assumed (of course because we do not know if they exist).

The full power of the stability result can only be exploited together with a comparison principle, as explained below: Assume we have an L^∞-bounded sequence of viscosity solutions to $G_\varepsilon(x, u, Du, D^2u) = 0$. A suitable comparison result of the limit equation shows $\overline{u} \leq \underline{u}$ on Ω or $\overline{\Omega}$; thus we can conclude that $\overline{u} = \underline{u}$ and u is actually continuous, and from the definition of $(\limsup)^*$, $(\liminf)_*$, one can prove that $(u_\varepsilon)_\varepsilon$ actually converges locally uniformly. For a continuous F, the stability result is of the type:

> Given a viscosity solution u^ε of an equation $F^\varepsilon(x, \varphi, D\varphi, D^2\varphi) = 0$, if $u^\varepsilon \to u$ and $F^\varepsilon \to F$ converge uniformly on compacts for $\varepsilon \to 0$, then u is a viscosity solution of $F(x, \varphi, D\varphi, D^2\varphi) = 0$.

The same principle as in a stability result is used for the proof of convergence of numerical schemes (see Barles and Souganidis [14]). We will discuss this for a special PIDE in §3.5.

On local properties. A short note on local properties, such as locally bounded, or locally uniform convergence: Let $\mathscr{O} \subset \mathbb{R}^d$ be an arbitrary set. A property holds locally in $\mathscr{O}$, if for each point $x \in \mathscr{O}$, there is a neighbourhood of x in $\mathscr{O}$ where this property holds. If one of the above properties holds locally in $\mathscr{O}$, then it holds on compacts $\subset \mathscr{O}$. For example, for a locally bounded function $u : \mathscr{O} \to \mathbb{R}$, as any open covering of a compact has a finite subcovering, u is bounded on this compact by the maximum of finitely many constants.

Chapter 2

Impulse control of jump-diffusions and viscosity solutions of HJBQVIs

We investigate in this chapter the relation between the value function of combined stochastic and impulse control and Hamilton-Jacobi-Bellman quasi-variational inequalities (HJBQVIs) in the weak sense of viscosity solutions (see Chapter 1).

The chapter consists of four main sections apart from the introduction. Section 2.2 presents the detailed problem formulation, the assumptions and a summary of the main result; substitutes for the dynamic programming principle are derived in §2.3. The following section is concerned with existence of a (discontinuous) viscosity solution of the HJBQVI. The last main section (§2.5) then deals with uniqueness: a maximum principle for impulse control is derived, and used in a comparison result, which yields uniqueness and continuity of the HJBQVI viscosity solution. An extension of the results to state-dependent intensity is discussed in §2.6.

The contents of this chapter correspond in large parts to the working paper Seydel [106], a shortened version of which was accepted for publication in "Stochastic Processes and their Applications" [107].

2.1 Introduction

Consider the combined stochastic and impulse control problem of the following SDE:

$$dX_t = \mu(t, X_{t-}, \beta_{t-})\, dt + \sigma(t, X_{t-}, \beta_{t-})\, dW_t + \int \ell(t, X_{t-}, \beta_{t-}, z)\, \overline{N}(dz, dt), \tag{2.1}$$

for a standard Brownian motion W and a compensated Poisson random measure $\overline{N}(dz, dt) = N(dz, dt) - 1_{|z|<1}\nu(dz)dt$ with possibly unbounded intensity measure ν (the jumps of a Lévy process), and the stochastic control process β (with values in some compact set B). The impulses occur at stopping times $(\tau_i)_{i\geq 1}$, and have the effect

$$X_{\tau_i} = \Gamma(t, \check{X}_{\tau_i-}, \zeta_i),$$

after which the process continues to evolve according to the controlled SDE until the next impulse. (Detailed notation and definitions are introduced in §2.2.) We denote by $\gamma = (\tau_i, \zeta_i)_{i\geq 1}$ the impulse control strategy, and by $\alpha = (\beta, \gamma)$ a combined control consisting of a stochastic control β and an impulse control γ. The aim is to maximize a certain functional, dependent on the impulse-controlled process X^α until the exit time τ (e.g., for a finite time horizon $T > 0$, $\tau := \tau_S \wedge T$, where τ_S is the exit time of X^α from a possibly unbounded set S):

$$v(t,x) := \max_\alpha \mathbb{E}^{(t,x)} \left[\int_t^\tau f(s, X_s^\alpha, \beta_s) ds + g(\tau, X_\tau^\alpha) 1_{\tau<\infty} + \sum_{\tau_j \le \tau} K(\tau_j, \check{X}_{\tau_j-}^\alpha, \zeta_j) \right] \tag{2.2}$$

Here the function K typically is negative and incorporates the impulse transaction costs, and the functions f and g are profit functions.

Quasi-variational inequality. The main purpose of this chapter is to prove that the value function v of (2.2) is the unique viscosity solution of the following partial integro-differential equation (PIDE), a so-called Hamilton-Jacobi-Bellman quasi-variational inequality (HJBQVI):

$$\min(-\sup_{\beta \in B}\{u_t + \mathcal{L}^\beta u + f^\beta\}, u - \mathcal{M}u) = 0 \qquad \text{in } [0,T) \times S, \tag{2.3}$$

together with suitable boundary conditions. Here, $\mathcal{L}^\beta$ is the infinitesimal generator of the SDE (2.1) (where $y = (t,x)$),

$$\begin{aligned} \mathcal{L}^\beta u(y) = \frac{1}{2} tr\left(\sigma(y,\beta)\sigma^T(y,\beta) D_x^2 u(y)\right) + \langle \mu(y,\beta), \nabla_x u(y) \rangle \\ + \int u(t, x + \ell(y,\beta,z)) - u(y) - \langle \nabla_x u(y), \ell(y,\beta,z) \rangle 1_{|z|<1}\, \nu(dz), \end{aligned}$$

and $\mathcal{M}$ is the intervention operator selecting the momentarily best impulse,

$$\mathcal{M}u(t,x) = \sup_\zeta \{u(t, \Gamma(t,x,\zeta)) + K(t,x,\zeta)\}.$$

(2.3) is formally a nonlinear, nonlocal, possibly degenerate, second order parabolic PIDE. We point out that the investigated stochastic process is allowed to have jumps (jump-diffusion process), including so-called "infinite-activity processes" where the jump measure ν may be singular at the origin. (It can be argued that infinite-activity processes are a good model for stock prices, see, e.g., Cont and Tankov [30], Eberlein and Keller [41].)

Solution approaches to impulse control. The standard approach to solve the impulse control problem (2.2) is certainly to analyze the QVI (2.3). This can be done using the theory of viscosity solutions as in this thesis; for an introduction and more references, see Chapter 1. In analogy with optimal stopping problems (or variational inequalities), the original approach advocated by Bensoussan and Lions [17] was to find weak L^p-solutions in the variational sense by a sequence of optimal stopping problems; see also the more recent paper Baccarin and Sanfelici [7].

Other techniques work directly with the associated dynamic programming equation in spaces of continuous functions (Palczewski and Zabczyk [95], Davis [37]), or consider the problem from a general Markov renewal theory perspective (Lepeltier and Marchal [77]). For an interesting variational approach, see Menaldi [82], [83]. Many approaches have in common that they approximate the problem by iterated optimal stopping; see Chapter 3 or Øksendal and Sulem [93] for details.

Impulse control & applications. The setting of our problem can be interesting for a number of applications, particularly in finance. Because impulse control problems typically involve fixed transaction costs — as opposed to singular control (only proportional costs), or stochastic control (no interventions) — they lend themselves readily to financial models in incomplete markets.

Clearly, the standard reference for applications as well as for theory is Bensoussan and Lions [17]; for a more recent overview for jump-diffusions, see Øksendal and Sulem [93]. For further applications in finance see the overview in Korn [72], or specific examples concerning option pricing with transaction costs ([110], [36], [21]), optimal portfolios ([71], [100], [95], [92]), options in long-term insurance contracts [29], or control of an exchange rate by the Central Bank ([87], [23]).

This last application is a good example for combined control: there are two different means of intervention, namely interest rates (stochastic control) and foreign exchange market interventions. The stochastic control affects the process continuously (we neglect transaction costs here), and the impulses have fixed transaction costs, but have an immediate effect and thus can better react to jumps in the stochastic process.

Our goal in writing this chapter was to establish a framework that can be readily used (and extended) in applications, without too many technical conditions.

Overview of the chapter. Main contribution of this chapter (and of the corresponding papers [106], [107]) is to rigorously treat viscosity solution existence and uniqueness of the HJBQVI (2.3) and of its elliptic counterpart, i.e., of the exit time problem for combined stochastic and impulse control of the jump-diffusion (2.1), whose jump part may have infinite activity.

Such a result is well known in the diffusion case (in a general setting, see Ishii [65], Tang and Yong [111]; for specific applications, see Ly Vath et al. [80], Øksendal and Sulem [92], Akian et al. [3]), and was established for piecewise deterministic processes (no exit time, and jumps with finite activity) in Lenhart [76] without stochastic control. To our best knowledge, there is no such result for jump-diffusion processes yet (even in the finite activity case and without stochastic control). A singular integral term complicates the problem considerably; we cater for this using techniques and results from Barles and Imbert [11].

Our general setting is probably closest to the one in the book [93] (where a sketch of proof for existence in the jump-diffusion case is offered, under the assumption that the value function is continuous); see also Pham [99] for optimal stopping and control until a finite time T. In the diffusion case, [111] prove existence and uniqueness of continuous viscosity solutions for a finite time horizon (no exit time) including stochastic control and optimal switching[1], but under rather restrictive assumptions. Besides, their approach requires continuity of the value function (which is proved on 11 pages). We note also that our problem (no exit time, and without stochastic control) was already treated in Menaldi [83] by non-viscosity solution techniques.

Let us now give a short overview on the methods we employ in this chapter to prove existence and uniqueness. We prove that the value function is a discontinuous viscosity solution of (2.3), as done in the recent paper [80] in a portfolio optimization context for a diffusion process. For the exit time problem, we need some continuity assumptions on the boundary ∂S — apart from that, the continuity will be a consequence of viscosity solution uniqueness. Because the jumps could lead outside S (and impulses could bring us back), we have to investigate the QVI on the whole space $\mathbb{R}^d$ with appropriate boundary conditions (the "boundary" is in general not a null

[1] Optimal switching can be considered as a special case of impulse control with higher-dimensional state space.

set of $\mathbb{R}^d$).
For the uniqueness proof, we use a perturbation technique with strict viscosity sub and supersolutions (as in Ishii [65]); this also takes care of the unboundedness of the domain. Our solutions can be unbounded at infinity with arbitrary polynomial growth (provided appropriate conditions on the functions involved are satisfied), and superlinear transaction costs (e.g., quadratic) are allowed.
Because the Lévy measure ν is allowed to have a singularity of second order at 0, we cannot use the standard approach to uniqueness of viscosity solutions of PDE as used in Pham [99] for optimal stopping, and in Benth et al. [18] for singular control. For a more detailed discussion, we refer to Jakobsen and Karlsen [68] (and the references therein), who were the first to propose a way to circumvent the problem for an HJB PIDE; see also the remark in the uniqueness section. For our proof of uniqueness, we will use and extend the framework as presented in the more recent paper Barles and Imbert [11] (the formulation in [68] does not permit an easy impulse control extension). The reader might also find helpful Barles et al. [15].

The chapter is organized as follows: Section 2.2 presents the detailed problem formulation, the assumptions and a summary of the main result; (substitutes for) the dynamic programming principle are derived in §2.3. The following §2.4 is concerned with existence of a HJBQVI viscosity solution. After introducing the setting of our impulse control problem and several helpful results, we prove in Theorem 2.4.2 that the value function is a (discontinuous) HJBQVI viscosity solution. The existence result for the elliptic HJBQVI is deduced from the corresponding parabolic one. The last main section (§2.5) then starts with a reformulation of the HJBQVI and several equivalent definitions of viscosity solutions. A maximum principle for impulse control is then derived, and used in a comparison result, which yields uniqueness and continuity of the HJBQVI viscosity solution. The chapter is complemented by an extension of the results to state-dependent intensity in §2.6.

2.2 Setting and main result

Let a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual assumptions be given. Consider an adapted m-dimensional Brownian motion W, and an adapted independent k-dimensional pure-jump Lévy process represented by the compensated Poisson random measure $\overline{N}(dz, dt) = N(dz, dt) - 1_{|z|<1}\nu(dz)dt$, where as always $\int(|z|^2 \wedge 1)\,\nu(dz) < \infty$ for the Lévy measure ν.[2] We assume as usual that all processes are right-continuous. Assume the d-dimensional state process X follows the stochastic differential equation with impulses

$$
\begin{aligned}
dX_t &= \mu(t, X_{t-}, \beta_{t-})\,dt + \sigma(t, X_{t-}, \beta_{t-})\,dW_t + \int_{\mathbb{R}^k} \ell(t, X_{t-}, \beta_{t-}, z)\,\overline{N}(dz, dt), \qquad \tau_i < t < \tau_{i+1} \\
X_{\tau_{i+1}} &= \Gamma(t, \check{X}_{\tau_{i+1}-}, \zeta_{i+1}), \qquad i \in \mathbb{N}_0,
\end{aligned}
\tag{2.4}
$$

for $\Gamma : \mathbb{R}_0^+ \times \mathbb{R}^{2d} \to \mathbb{R}^d$ measurable, and $\mu : \mathbb{R}_0^+ \times \mathbb{R}^d \times B \to \mathbb{R}^d$, $\sigma : \mathbb{R}_0^+ \times \mathbb{R}^d \times B \to \mathbb{R}^{d\times m}$, $\ell : \mathbb{R}_0^+ \times \mathbb{R}^d \times B \times \mathbb{R}^k \to \mathbb{R}^d$ satisfying the necessary conditions such that existence and uniqueness of the SDE is guaranteed. β is a càdlàg adapted stochastic control (where $\beta(t,\omega) \in B$, B compact non-empty metric space), and $\gamma = (\tau_1, \tau_2, \ldots, \zeta_1, \zeta_2, \ldots)$ is the impulse control strategy, where τ_i are stopping times with $0 = \tau_0 \leq \tau_1 \leq \tau_2 \leq \ldots$, and ζ_i are adapted impulses. The measurable

[2] Note that the normal definition of a compensator is with the indicator function $1_{|z|\leq 1}$; this is however an equivalent formulation.

transaction set $Z(t,x) \subset \mathbb{R}^d$ denotes the allowed impulses when at time t in state x. We denote by $\alpha = (\beta, \gamma)$ the combined stochastic and impulse control, where $\alpha \in \mathcal{A} = \mathcal{A}(t,x)$, the admissible set for the combined stochastic and impulse control. Admissible means here apart from the above that existence and uniqueness of the SDE be guaranteed, and that we only consider Markov controls (detailed in §2.3). We further assume that all constant stochastic controls β in B are admissible.

The term $\check{X}^\alpha_{\tau_j-}$ denotes the value of the controlled process X^α in τ_j with a possible jump of the stochastic process, but without the impulse, i.e., $\check{X}^\alpha_{\tau_j-} = X^\alpha_{\tau_j-} + \Delta X^\alpha_{\tau_j}$, where Δ denotes the jump of the stochastic process. So for the first impulse, this would be the process $\check{X}^\alpha_{\tau_1-} = X^\beta_{\tau_1}$ only controlled by the continuous control. If two or more impulses happen to be at the same time (e.g., $\tau_{i+1} = \tau_i$), then (2.4) is to be understood as concatenation, e.g., $\Gamma(t, \Gamma(t, \check{X}_{\tau_i-}, \zeta_i), \zeta_{i+1})$. (The notation used here is borrowed from Øksendal and Sulem [93].)

The following conditions are sufficient for existence and uniqueness of the SDE (2.4) for constant stochastic control β (cf. Gikhman and Skorokhod [53], p. 273): There is an $x \in \mathbb{R}^d$ and $C > 0$ with $\int |\ell(t,x,\beta,z)|^2 \nu(dz) \le C < \infty$ for all $t \in [0,T]$, $\beta \in B$. Furthermore, there exist $C > 0$ and a positive function b with $b(h) \downarrow 0$ for $h \to 0$ s.th.

$$
\begin{aligned}
&|\mu(t,x,\beta) - \mu(t,y,\beta)|^2 + |\sigma(t,x,\beta) - \sigma(t,y,\beta)|^2 \\
&\qquad + \int |\ell(t,x,\beta,z) - \ell(t,y,\beta,z)|^2 \nu(dz) \le C|x-y|^2 \\
&|\mu(t+h,x,\beta) - \mu(t,x,\beta)|^2 + |\sigma(t+h,x,\beta) - \sigma(t,x,\beta)|^2 \\
&\qquad + \int |\ell(t+h,x,\beta,z) - \ell(t,x,\beta,z)|^2 \nu(dz) \le C(1+|x|^2) b(h).
\end{aligned}
\tag{2.5}
$$

These are essentially Lipschitz conditions, as one can easily check; see also Pham [99]. A solution of (2.4) with non-random starting value will then have finite second moments, which is preserved after an impulse if for $X \in L^2(\mathbb{P};\mathbb{R}^d)$, also $\Gamma(t, X, \zeta(t,X)) \in L^2(\mathbb{P};\mathbb{R}^d)$. This is certainly the case if the impulses ζ_j are in a compact and Γ continuous, which we shall assume later.
We only assume that existence and uniqueness hold with constant stochastic control (by (2.5) or weaker conditions as in [53]), which guarantees that $\mathcal{A}(t,x)$ is non-empty.

Remark 2.2.1. For existence and uniqueness of the SDE with arbitrary control process β, it is in general not sufficient to simply assume Lipschitz conditions on $x \mapsto \mu(t,x,\beta)$ etc. (even if the Lipschitz constants are independent of the control β). If the control depends in a non-Lipschitzian or even discontinuous way on the current state, uniqueness or even existence of (2.4) might not hold. Compare also Gikhman and Skorokhod [54], p. 156.

Remark 2.2.2. The version of the SDE in (2.4) is the most general form of all Lévy SDE formulations currently used. If the support of ν is contained in the coordinate axes, then the one-dimensional components of the Lévy process are independent, and the integral $\int_{\mathbb{R}^k} \ell(t, X_{t-}, \beta_{t-}, z)$ $\overline{N}(dz, dt)$ can be splitted into several one-dimensional integrals (i.e., we obtain the form used in Øksendal and Sulem [93]). If furthermore ℓ is linear in z, then we obtain the form as used in Protter [101], Th. V.32.

The general (combined) stochastic and impulse control problem is: find $\alpha = (\beta, \gamma) \in \mathcal{A}$ that maximizes the payoff starting in t with x

$$
J^{(\alpha)}(t,x) = \mathbb{E}^{(t,x)} \left[\int_t^\tau f(s, X^\alpha_s, \beta_s) ds + g(\tau, X^\alpha_\tau) 1_{\tau < \infty} + \sum_{\tau_j \le \tau} K(\tau_j, \check{X}^\alpha_{\tau_j-}, \zeta_j) \right], \tag{2.6}
$$

where $f : \mathbb{R}_0^+ \times \mathbb{R}^d \times B \to \mathbb{R}$, $g : \mathbb{R}_0^+ \times \mathbb{R}^d \to \mathbb{R}$, $K : \mathbb{R}_0^+ \times \mathbb{R}^{2d} \to \mathbb{R}$ are measurable, and $\tau = \tau_S = \inf\{s \geq t : X_s^\alpha \notin S\}$ is the exit time from some open set $S \subset \mathbb{R}^d$ (possibly infinite horizon), or $\tau = \tau_S \wedge T$ for some $T > 0$ (finite horizon). Note that X_s^α is the value at s after all impulses in s have been applied; so "intermediate values" are not taken into account by this stopping time.

The value function v of combined stochastic and impulse control is defined by

$$v(t,x) = \sup_{\alpha \in \mathcal{A}(t,x)} J^{(\alpha)}(t,x) \tag{2.7}$$

We require the integrability condition on the negative parts of f, g, K

$$\mathbb{E}^{(t,x)}\left[\int_t^\tau f^-(s, X_s^\alpha, \beta_s)ds + g^-(\tau, X_\tau^\alpha)1_{\tau<\infty} + \sum_{\tau_j \leq \tau} K^-(\tau_j, \check{X}^\alpha_{\tau_j-}, \zeta_j)\right] < \infty \tag{2.8}$$

for all $\alpha \in \mathcal{A}(t,x)$.

Parabolic HJBQVI. For a fixed finite horizon $T > 0$, we define $S_T := [0,T) \times S$ and its parabolic nonlocal "boundary" $\partial^+ S_T := \big([0,T) \times (\mathbb{R}^d \setminus S)\big) \cup \big(\{T\} \times \mathbb{R}^d\big)$. Further denote $\partial^* S_T := ([0,T) \times \partial S) \cup \big(\{T\} \times \overline{S}\big)$. Let the impulse intervention operator $\mathcal{M} = \mathcal{M}^{(t,x)}$ be defined by

$$\mathcal{M}u(t,x) = \sup\{u(t, \Gamma(t,x,\zeta)) + K(t,x,\zeta) : \zeta \in Z(t,x)\} \tag{2.9}$$

(define $\mathcal{M}u(t,x) = -\infty$ if $Z(t,x) = \emptyset$ – we will exclude this case later on). The hope is to find the value function by investigating the following parabolic Hamilton Jacobi Bellman QVI:

$$\begin{aligned} \min(-\sup_{\beta \in B}\{u_t + \mathcal{L}^\beta u + f^\beta\}, u - \mathcal{M}u) &= 0 \qquad \text{in } S_T \\ \min(u - g, u - \mathcal{M}u) &= 0 \qquad \text{in } \partial^+ S_T, \end{aligned} \tag{2.10}$$

for $\mathcal{L}^\beta$ the generator of X in the SDE (2.4) for constant stochastic control β, and $f^\beta(\cdot) := f(\cdot, \beta)$. The generator $\mathcal{L}^\beta$ has the form $(y = (t,x))$:

$$\begin{aligned} \mathcal{L}^\beta u(y) = &\frac{1}{2} tr\left(\sigma(y,\beta)\sigma^T(y,\beta)D_x^2 u(y)\right) + \langle \mu(y,\beta), \nabla_x u(y)\rangle \\ &+ \int u(t, x + \ell(y,\beta,z)) - u(y) - \langle \nabla_x u(y), \ell(y,\beta,z)\rangle 1_{|z|<1}\, \nu(dz). \end{aligned} \tag{2.11}$$

An intuitive explanation of (2.10) in S_T would be as follows (without stochastic control): Either the stochastic process can evolve according to the SDE, and then by Itô's formula $v_t + \mathcal{L}v + f = 0$, or it is optimal to give an impulse, i.e., $v = \mathcal{M}v$. In any case $v(t, X_t^\alpha)$ is a supermartingale, and impulses cannot improve the value function v because it is already optimal, i.e. $v \geq \mathcal{M}v$.

We still have to argue why we consider the value function v on $[0,T) \times \mathbb{R}^d$ instead of the interesting set $[0,T) \times S$: This is due to the jump term of the underlying stochastic process. While it is not possible to stay a positive time outside S (we stop in τ_S), it is well possible in our setting that the stochastic process jumps outside, but we return to S by an impulse before the stopping time τ_S takes notice. Thus we must define v outside S, to be able to decide whether a jump back to S is worthwile. The boundary condition has its origin in the following necessary condition for the value function:

$$\min(v - g, v - \mathcal{M}v) = 0 \qquad \text{in } [0,T) \times (\mathbb{R}^d \setminus S) \tag{2.12}$$

This formalizes that the controller can either do nothing (i.e., at the end of the day, the stopping time τ_S has passed, and the game is over), or can jump back into S, and the game continues. A similar condition holds at time $T < \infty$, with the difference that the controller is not allowed to jump back in time (as the device permitting this is not yet available to the public). So the necessary terminal condition can be put explicitly as[3]

$$v = \sup(g, \mathcal{M}g, \mathcal{M}^2 g, \ldots) \qquad \text{on } \{T\} \times \mathbb{R}^d. \tag{2.13}$$

Example 2.2.1. The impulse back from $\mathbb{R}^d \setminus S$ to S could correspond to a capital injection into profitable business to avoid untimely default due to a sudden event.

Well-definedness of QVI terms. We need to establish conditions under which the terms $\mathcal{L}^\beta u$ and $\mathcal{M}u$ in the QVI (2.10) are well-defined. For $\mathcal{M}u$ compare the discussion of assumptions below.

The integral operator in (2.11) is at the same time a differential operator of up to second order (if ν is singular). This can be seen by Taylor expansion for $u \in C^{1,2}([0,T)\times\mathbb{R}^d)$ for some $0 < \delta < 1$:

$$\begin{aligned} &\int_{|z|<\delta} |u(t, x + \ell(x,\beta,z)) - u(t,x) - \langle \nabla u(t,x), \ell(x,\beta,z)\rangle 1_{|z|<1}|\, \nu(dz) \\ &\leq \int_{|z|<\delta} |\ell(t,x,\beta,z)|^2 |D^2 u(t,\tilde{x})|\, \nu(dz) \end{aligned} \tag{2.14}$$

for an $\tilde{x} \in B(x, \sup_{|z|<\delta} \ell(t,x,\beta,z))$.

We take the Lévy measure ν as given. As for all Lévy measures, $\int (|z|^2 \wedge 1)\, \nu(dz) < \infty$. Let $p^* > 0$ be a number such that $\int_{|z|\geq 1} |z|^{p^*}\, \nu(dz) < \infty$, and let $q^* \geq 0$ be a number such that $\int_{|z|<1} |z|^{q^*}\, \nu(dz) < \infty$ (think of p^* as the *largest* and q^* the *smallest* such number, even if it does not exist). Then the expression $\sup_{\beta\in B}\{u_t + \mathcal{L}^\beta u + f^\beta\}$ is well-defined, if, e.g., the following conditions are satisfied (depending on the singularity of ν in 0 and its behaviour at infinity):

1. $u \in C^{1,2}([0,T)\times\mathbb{R}^d)$, $\sup_{\beta\in B} \sigma(t,x,\beta) < \infty$, $\sup_{\beta\in B} \mu(t,x,\beta) < \infty$, and $\sup_{\beta\in B} f(t,x,\beta) < \infty$ for all $(t,x) \in [0,T) \times S$
2. One of the following (all constants are independent of β, z, and the inequalities hold locally in $(t,x) \in [0,T) \times S$ if $C_{t,x}$ is used as constant[4]):
   (a) $|\ell(t,x,\beta,z)| \leq C_{t,x}$ if $\nu(\mathbb{R}^d) < \infty$
   (b) $|\ell(t,x,\beta,z)| \leq C_{t,x}(|z|)$ and $|u(t,z)| \leq C(1 + |z|^{p^*})$ (C independent of t)
   (c) Or more generally, for $a, b > 0$ such that $ab \leq p^*$: $|\ell(t,x,\beta,z)| \leq C_{t,x}(1 + |z|^a)$, $|u(t,z)| \leq C(1 + |z|^b)$ on $|z| \geq 1$. Furthermore, $|\ell(t,x,\beta,z)| \leq C_{t,x}|z|^{q^*/2}$ on $|z| < 1$

In the following, we will assume one of the above conditions on ℓ (which is given), and define as in Barles and Imbert [11] for a fixed polynomial function $R : \mathbb{R}^d \to \mathbb{R}$ satisfying condition 2.:

[3] If g is lower semicontinuous, if $\mathcal{M}$ preserves this property and if the sup is finite, then it is well known that $v(T,\cdot)$ is lower semicontinuous. Even if g is continuous, $v(T,\cdot)$ need not be continuous.

[4] I.e., for all $(t,x) \in [0,T) \times S$, there exists a constant $C_{t,x} > 0$ and a neighbourhood of (t,x) in $[0,T) \times S$ such that $|\ell(t,x,\beta,z)| \leq C_{t,x}$ for all β, z.

Definition 2.2.1 (Space of polynomially bounded functions)**.** *$\mathcal{PB} = \mathcal{PB}([0,T) \times \mathbb{R}^d)$ is the space of all measurable functions $u : [0,T) \times \mathbb{R}^d \to \mathbb{R}$ such that*

$$|u(t,x)| \leq C_u(1 + R(x))$$

for a time-independent constant $C_u > 0$.

As pointed out in [11], this function space $\mathcal{PB}$ is stable under lower and upper semicontinuous envelopes, and functions in $\mathcal{PB}$ are locally bounded. Furthermore, it is stable under (pointwise) limit operations, and the conditions for Lebesgue's dominated convergence theorem are in general satisfied.

Assumptions and main result. Let us now formalize the conditions necessary for both the existence and the uniqueness proof in the following assumption (see also the discussion at the end of the section):

Assumption 2.2.1. *(V1) Γ and K are continuous.*

(V2) The transaction set $Z(t,x)$ is non-empty and compact for each $(t,x) \in [0,T] \times \mathbb{R}^d$. For a converging sequence $(t_n,x_n) \to (t,x)$ in $[0,T] \times \mathbb{R}^d$ (with $Z(t_n,x_n)$ non-empty), $Z(t_n,x_n)$ converges to $Z(t,x)$ in the Hausdorff metric.

(V3) μ, σ, ℓ, f are continuous in (t,x,β) on $[0,T) \times \mathbb{R}^d \times B$.

(V4) ℓ satisfies one of the conditions detailed above, and $\mathcal{PB}$ is fixed accordingly.

Apart from Assumption 2.2.1, we will need the following assumptions for the proof of existence:

Assumption 2.2.2. *(E1) The value function v is in $\mathcal{PB}([0,T] \times \mathbb{R}^d)$.*

(E2) g is continuous.

(E3) The value function v satisfies for every $(t,x) \in \partial^* S_T$, and all sequences $(t_n,x_n)_n \subset [0,T) \times S$ converging to (t,x) with $v(t_n,x_n) \to z \in \mathbb{R}$:

$$\begin{aligned} & z \geq g(t,x) \\ \text{If } v^*(t,x) > \mathcal{M}v^*(t,x) : \quad & z \leq g(t,x), \end{aligned}$$

where v^* is the upper semicontinuous envelope of v on $[0,T] \times \mathbb{R}^d$.

(E4) For all $\rho > 0$, $(t,x) \in [0,T) \times S$, and sequences $[0,T) \times S \ni (t_n,x_n) \to (t,x)$, there is a constant $\hat\beta$ (not necessarily in B) and an $N \in \mathbb{N}$ such that for all $0 < \varepsilon < 1/N$ and $n \geq N$,

$$\mathbb{P}\Big(\sup_{t_n \leq s \leq t_n+\varepsilon} |X_s^{\hat\beta,t_n,x_n} - x_n| < \rho\Big) \leq \mathbb{P}\Big(\sup_{t_n \leq s \leq t_n+\varepsilon} |X_s^{\beta^n,t_n,x_n} - x_n| < \rho\Big),$$

where X^{β^n,t_n,x_n} is the process according to SDE (2.4), started in (t_n,x_n) and controlled by β^n.

The following assumptions are needed for the uniqueness proof ($\delta > 0$):

Assumption 2.2.3. *(U1) If $\nu(\mathbb{R}^k) = \infty$: For all $(t,x) \in [0,T) \times S$, $U_\delta(t,x) := \{\ell(x,\beta,z) : |z| < \delta\}$ does not depend on β.*

(U2) If $\nu(\mathbb{R}^k) = \infty$: For all $(t,x) \in [0,T) \times S$, $dist(x, \partial U_\delta(t,x))$ is strictly positive for all $\delta > 0$ (or $\ell \equiv 0$).

Assumption 2.2.4. *(B1) $\int_{\mathbb{R}^d} |\ell(t,x,\beta,z) - \ell(t,y,\beta,z)|^2 \nu(dz) < C|x-y|^2$, $\int_{|z|\geq 1} |\ell(t,x,\beta,z) - \ell(t,y,\beta,z)|\nu(dz) < C|x-y|$, and all estimates hold locally in $t \in [0,T)$, x, y, uniformly in β.*

(B2) Let $\sigma(\cdot,\beta)$, $\mu(\cdot,\beta)$, $f(\cdot,\beta)$ be locally Lipschitz continuous, i.e. for each point $(t_0,x_0) \in [0,T) \times S$ there is a neigbourhood $U \ni (t_0,x_0)$ (U open in $[0,T) \times \mathbb{R}^d$), and a constant C (independent of β) such that $|\sigma(t,x,\beta) - \sigma(t,y,\beta)| \leq C|x-y|$ $\forall (t,x),(t,y) \in U$, and likewise for μ and f.

The space $\mathcal{PB}_p = \mathcal{PB}_p([0,T] \times \mathbb{R}^d)$ consists of all functions $u \in \mathcal{PB}$, for which there is a constant C such that $|u(t,x)| \leq C(1+|x|^p)$ for all $(t,x) \in [0,T] \times \mathbb{R}^d$.

Under the above assumptions, we can now formulate our main result in the parabolic case (the precise definition of viscosity solution is introduced in Section 2.4):

Theorem 2.2.2 (Viscosity existence and uniqueness for parabolic HJBQVI)**.** *Let Assumptions 2.2.1-2.2.4 be satisfied. Assume further that $v \in \mathcal{PB}_p([0,T] \times \mathbb{R}^d)$, and that there is a nonnegative $w \in \mathcal{PB} \cap C^{1,2}([0,T] \times \mathbb{R}^d)$ with $w(t,x)/|x|^p \to \infty$ for $|x| \to \infty$ (uniformly in t) and a constant $\kappa > 0$ such that*

$$\begin{aligned} \min(-\sup_{\beta \in B}\{w_t + \mathcal{L}^\beta w + f^\beta\}, w - \mathcal{M}w) &\geq \kappa \qquad \text{in } S_T \\ \min(w-g, w-\mathcal{M}w) &\geq \kappa \qquad \text{in } \partial^+ S_T. \end{aligned}$$

Then the value function v in (2.7) is the unique viscosity solution in $\mathcal{PB}_p([0,T] \times \mathbb{R}^d)$ of the parabolic HJBQVI (2.10), and it is continuous on $[0,T] \times \mathbb{R}^d$.

For the proof of Theorem 2.2.2, see Theorems 2.4.2 and 2.5.14.

Elliptic HJBQVI. For finite time horizon T, (2.10) is investigated on $[0,T] \times \mathbb{R}^d$ (parabolic problem). For infinite horizon, typically a discounting factor $e^{-\rho(t+s)}$ for $\rho > 0$ applied to the functions f, g and K takes care of the well-definedness of the value function, e.g., $f(t,x,\beta) = e^{-\rho(t+s)}\tilde{f}(x,\beta)$; furthermore, the SDE coefficients in (2.4) must be time-independent. In this time-independent case, a transformation $u(t,x) = e^{-\rho(t+s)}\tilde{u}(x)$ gives us the elliptic HJBQVI to investigate

$$\begin{aligned} \min(-\sup_{\beta \in B}\{-\rho u + \mathcal{L}^\beta u + f^\beta\}, u - \mathcal{M}u) &= 0 \qquad \text{in } S \\ \min(u-g, u-\mathcal{M}u) &= 0 \qquad \text{in } \mathbb{R}^d \setminus S, \end{aligned} \tag{2.15}$$

where the functions and variables have been appropriately renamed, and

$$\begin{aligned} \mathcal{L}^\beta u(x) = \frac{1}{2} tr\left(\sigma(x,\beta)\sigma^T(x,\beta)D^2u(x)\right) &+ \langle \mu(x,\beta), \nabla u(x)\rangle \\ &+ \int u(x+\ell(x,\beta,z)) - u(x) - \langle \nabla u(x), \ell(x,\beta,z)\rangle 1_{|z|<1}\, \nu(dz), \end{aligned} \tag{2.16}$$

$$\mathcal{M}u(x) = \sup\{u(\Gamma(x,\zeta)) + K(x,\zeta) : \zeta \in Z(x)\}. \tag{2.17}$$

Under the time-independent version of the assumptions above, an essentially identical existence and uniqueness result holds for the elliptic HJBQVI (2.15). We refrain from repeating it, and instead refer to Sections 2.4.2 and 2.5.4 for a precise formulation.

Discussion of the assumptions. Of all assumptions, it is quite clear why we need the continuity assumptions, and they are easy to check.

By (V3), (V4) and the compactness of the control set B, the Hamiltonian $\sup_{\beta \in B} u_t(t,x) + \mathcal{L}^\beta u(t,x) + f^\beta(t,x)$ is well-defined and continuous in $(t,x) \in [0,T) \times S$ for $u \in \mathcal{PB} \cap C^{1,2}([0,T) \times \mathbb{R}^d)$. (This follows by sup manipulations, the (locally) uniform continuity and the DCT for the integral part.) Instead of (V3), assuming the continuity of the Hamiltonian is sufficient for the existence proof. For the stochastic process X_t, condition (V4) essentially ensures the existence of moments.

By (V1), (V2), we obtain that $\mathcal{M}u$ is locally bounded if u is locally bounded in $[0,T] \times \mathbb{R}^d$ (e.g., if $u \in \mathcal{PB}([0,T] \times \mathbb{R}^d)$). $\mathcal{M}u$ is even continuous if u is continuous (so impulses preserve continuity properties), see Lemma 2.4.3.

In condition (V2), $Z(t,x) \neq \emptyset$ is necessary for the Hausdorff metric of sets to be well-defined, and to obtain general results on continuity of the value function (it is easy to construct examples of discontinuous value functions otherwise). The assumption is however not a severe restriction, because we can set $Z(t,x) = \emptyset$ in the no-intervention region $\{v > \mathcal{M}v\}$ without affecting the value function. The compactness of $Z(t,x)$ is not essential and can be relaxed in special cases — this restriction is however of no practical importance.

Condition (E3) connects the combined control problem with the continuity of the stochastic control problem at the boundary. In this respect, Theorem 2.2.2 roughly states that the value function is continuous except if there is a discontinuity at the boundary ∂S. (E3) is typically satisfied if the stochastic process is regular at ∂S, as shown at the end of the §2.4.1; see also Fleming and Soner [46], Theorem V.2.1, and the analytical approach in Barles et al. [15], Barles and Rouy [13]. In particular, this condition excludes problems with true or *de facto* state constraints, although the framework can be extended to cover state constraints.

(E4) can be expected to hold because the control set B is compact and the functions μ, σ, ℓ are continuous in (t,x,β) (V3). The condition is very easy to check for a concrete problem — it would be a lot more cumbersome to state a general result, especially for the jump part.

Example 2.2.2. If $dX_t = \beta_t dt + dW_t$, and $\beta_t \in B = [-1,1]$, then $\hat{\beta} := 2$ is a possible choice for (E4) to hold.

Assumption 2.2.3 collects some minor prerequisites that only need to be satisfied for small $\delta > 0$ (see also the remark in the beginning of §2.5.1), and the formulation can easily be adapted to a specific problem.

The local Lipschitz continuity in (B1) and (B2) is a standard condition; (B1) is satisfied if, e.g., the jump size of the stochastic process does not depend on x, or the conditions (2.5) for existence and uniqueness of the SDE are satisfied for a constant β. Condition (B1) can be relaxed if, e.g., X has a state-dependent (finite) jump intensity; see §2.6 for details.

Certainly the most intriguing point is how to find a suitable function w meeting all the requirements detailed in Theorem 2.2.2. (This requirement essentially means that we have a strict supersolution.) We first consider the elliptic case of the HJBQVI (2.15). Here, such a function w for a $\kappa > 0$ can normally be constructed by $w(x) = w_1|x|^q + w_2$ for suitable w_i and $q > p$ (but still $w \in \mathcal{PB}$!). Main prerequisites are then

(L1) Positive interest rates: $\rho > \tilde{\kappa}$ for a suitably chosen constant $\tilde{\kappa} > 0$

(L2) Fixed transaction costs: e.g., $K(x,\zeta) \leq -k_0 < 0$

If additionally we allow only impulses towards 0, then $w - \mathcal{M}w > \kappa$ is easily achieved, as well as $w - g > \kappa$ (if we require that g have a lower polynomial order than w). For a given bounded set, choosing w_2 large enough makes sure that $-\sup_{\beta \in B}\{-\rho w + \mathcal{L}^\beta w + f^\beta\} > \kappa$ on this set (due to the continuity of the Hamiltonian and translation invariance in the integral). For $|x| \to \infty$, we need to impose conditions on $\tilde{\kappa}$ — these depend greatly on the problem at hand, but can require the discounting factor to be rather large (e.g., for a geometric Brownian motion).

In the parabolic case, the same discussion holds accordingly, except that it is significantly easier to find a $w \in \mathcal{PB}([0,T] \times \mathbb{R}^d)$ satisfying assumption (L1): By setting $w(t,x) = \exp(-\tilde{\kappa}t)(w_1|x|^q + w_2)$, we have $w_t = -\tilde{\kappa}w$ for arbitrarily large $\tilde{\kappa}$.

2.3 Probabilistic properties

In this section, we establish the Markov property of the impulse-controlled process, and derive a version of the dynamic programming principle (DPP) which we will need for the existence proof of §2.4.

Markov property. For the Dynkin formula and several transformations, we need to establish the Markov property of the controlled process X^α. To be more precise, we need to prove the strong Markov property of $Y_t^\alpha := (s+t, X_{s+t}^\alpha)$ for some $s \geq 0$. First, by Gikhman and Skorokhod [53] Theorem 1 in Part II.9, the Markov property for the uncontrolled process X holds (for a similar result, see Protter [101], Th. V.32). If we consider the process $Y_t := (s+t, X_{s+t})$, then the strong Markov property holds for Y.

We only consider Markov controls in the following, i.e. $\beta(t,\omega) = \beta(Y^\alpha(t-))$, the impulse times τ_i are exit times of Y_t^α (which makes sure they do not use past information), and ζ_i are $\sigma(\check{Y}_{\tau_i-}^\alpha)$-measurable. That Markov controls are sufficient is clear intuitively, because it is of no use to consider the past, if my objective function only depends on future actions and events, and my underlying process already has the Markov property.

Proposition 2.3.1. *Under the foregoing assumptions, the controlled process Y^α is a strong Markov process.*

Proof: For a stopping time $T < \infty$ a.s., we have to show for all bounded measurable functions h and for all $u \geq 0$ that $\mathbb{E}^y[h(Y_{T+u}^\alpha)|\mathcal{F}_T]$ is actually Y_T^α-measurable. First note that without impulses, because $\beta(t-,\omega) = \beta(Y^\alpha(t-))$, the SDE solution Y^β has the strong Markov property by the above cited results; we denote by $Y^\beta(y,t,t+u)$ this SDE solution started at $t \geq 0$ in y, evaluated at $t+u$.

Wlog, all $\tau_i \geq T$, e.g., τ_1 first exit time after T. We split into the cases $A_0 := \{\tau_1 > T+u\}$, $A_1 := \{\tau_1 \leq T+u < \tau_2\}$, $A_2 := \{\tau_2 \leq T+u < \tau_3\}$, ... The case $\tau_1 > T+u$ is clear by the above. For A_1,

$$\begin{aligned}
\mathbb{E}^y[1_{A_1}h(Y_{T+u}^\alpha)|\mathcal{F}_T] &= \mathbb{E}^y[1_{A_1}\mathbb{E}^y[h(Y_{T+u}^\alpha)|\mathcal{F}_{\tau_1}]|\mathcal{F}_T] \\
&= \mathbb{E}^y[1_{A_1}\mathbb{E}^y[h(Y^\beta(\Gamma(\check{Y}_{\tau_1-}^\alpha,\zeta_1),\tau_1,T+u-\tau_1))|\mathcal{F}_{\tau_1}]|\mathcal{F}_T].
\end{aligned}$$

Because τ_1 is the first exit time after T (and thus $T+u-\tau_1$ independent of T), and $Y_{\tau_1-}^\alpha$ includes the time information τ_1 as first component, the SDE solution $Y^\beta(\Gamma(\check{Y}_{\tau_1-}^\alpha,\zeta_1),\tau_1,T+u-\tau_1)$ depends only on $\check{Y}_{\tau_1-}^\alpha$. Thus we can conclude that there are measurable functions $g_1, \tilde{g}_1$ such that

$$\mathbb{E}^y[1_{A_1}h(Y_{T+u}^\alpha)|\mathcal{F}_T] = \mathbb{E}^y[1_{A_1}g_1(\check{Y}_{\tau_1-}^\alpha)|\mathcal{F}_T] = \mathbb{E}^y[1_{A_1}g_1((\check{Y}_{T+u-}^\alpha)^{\tau_1})|\mathcal{F}_T] = \tilde{g}_1(Y_T^\alpha),$$

where $(\check{Y}^{\alpha}_{T+u-})^{\tau_1}$ is the process stopped in τ_1, which is a strong Markov process by Dynkin [39], Th. 10.2 and the fact that τ_1 is the first exit time.

For A_2, following exactly the same arguments, there are measurable functions $g_i, \tilde{g}_i$ such that

$$\begin{aligned} \mathbb{E}^y[1_{A_1} h(Y^{\alpha}_{T+u})|\mathcal{F}_T] &= \mathbb{E}^y[1_{A_1} g_2(\check{Y}^{\alpha}_{\tau_2-})|\mathcal{F}_T] \\ &= \mathbb{E}^y[1_{A_1}\mathbb{E}^y[g_2(Y^{\beta}(\Gamma(\check{Y}^{\alpha}_{\tau_1-},\zeta_1),\tau_1,\tau_2-\tau_1))|\mathcal{F}_{\tau_1}]|\mathcal{F}_T] \\ &= \mathbb{E}^y[1_{A_1} g_1((\check{Y}^{\alpha}_{T+u-})^{\tau_1})|\mathcal{F}_T] = \tilde{g}_1(Y^{\alpha}_T), \end{aligned}$$

where we have used that τ_2 is the first exit time after τ_1. The result follows by induction and the dominated convergence theorem. □

Dynamic Programming Principle. An important insight into the structure of the problem is provided by Bellman's dynamic programming principle (DPP). Although the DPP is frequently used (see, e.g., Øksendal and Sulem [93], Ly Vath et al. [80]), we are only aware of the proof by Tang and Yong [111] in the impulse control case (for diffusions). We show here how the DPP can formally be derived from the Markov property.

By the strong Markov property of the controlled process, we have for a stopping time $\tilde{\tau}$ with $t \le \tilde{\tau} \le \tau$ ($\tau = \tau_S$ or $\tau = \tau_S \wedge T$):

$$\begin{aligned} J^{(\alpha)}(t,x) &= \mathbb{E}^{(t,x)}\Bigg\{\int_t^{\tilde{\tau}} f(s,X^{\alpha}_s,\beta_s)ds + \sum_{\tau_j<\tilde{\tau}} K(\tau_j,\check{X}^{\alpha}_{\tau_j-},\zeta_j) \\ &\quad + \mathbb{E}^{(\tilde{\tau},\check{X}^{\alpha}_{\tilde{\tau}-})}\Bigg[\int_{\tilde{\tau}}^{\tau} f(s,X^{\alpha}_s,\beta_s)ds + g(\tau,X^{\alpha}_{\tau})1_{\tau<\infty} + \sum_{\tilde{\tau}\le\tau_j\le\tau} K(\tau_j,\check{X}^{\alpha}_{\tau_j-},\zeta_j)\Bigg]\Bigg\} \\ &= \mathbb{E}^{(t,x)}\Bigg\{\int_t^{\tilde{\tau}} f(s,X^{\alpha}_s,\beta_s)ds + \sum_{\tau_j<\tilde{\tau}} K(\tau_j,\check{X}^{\alpha}_{\tau_j-},\zeta_j) + J^{(\alpha)}(\tilde{\tau},\check{X}^{\alpha}_{\tilde{\tau}-})\Bigg\} \qquad (2.18) \\ &\le \mathbb{E}^{(t,x)}\Bigg\{\int_t^{\tilde{\tau}} f(s,X^{\alpha}_s,\beta_s)ds + \sum_{\tau_j<\tilde{\tau}} K(\tau_j,\check{X}^{\alpha}_{\tau_j-},\zeta_j) + v(\tilde{\tau},\check{X}^{\alpha}_{\tilde{\tau}-})\Bigg\}. \qquad (2.19) \end{aligned}$$

Note especially that the second J in (2.18) "starts" from $\check{X}_{\tilde{\tau}-}$, i.e. from X before applying the possible impulses in $\tilde{\tau}$ – this is to avoid counting a jump twice. $X_{\tilde{\tau}}$ instead of $\check{X}_{\tilde{\tau}-}$ in (2.18) would be incorrect (even if we replace the $=$ by a $\le$). However, $J^{(\alpha)}(\tilde{\tau},\check{X}_{\tilde{\tau}-}) \le v(\tilde{\tau},X_{\tilde{\tau}}) + K(\tilde{\tau},\check{X}_{\tilde{\tau}-},\zeta)1_{\{impulse\ in\ \tilde{\tau}\}}$ holds because a (possibly non-optimal) decision to give an impulse ζ in $\tilde{\tau}$ influences J and v in the same way. So we have the modified inequality

$$J^{(\alpha)}(t,x) \le \mathbb{E}^{(t,x)}\Bigg[\int_t^{\tilde{\tau}} f(s,X^{\alpha}_s,\beta_s)ds + \sum_{\tau_j\le\tilde{\tau}} K(\tau_j,\check{X}^{\alpha}_{\tau_j-},\zeta_j) + v(\tilde{\tau},X^{\alpha}_{\tilde{\tau}})\Bigg]. \qquad (2.20)$$

We will use both inequalities in the proof of Theorem 2.4.2 (viscosity existence).

The above considerations can be formalized in the well known dynamic programming principle (DPP) (if the admissibility set $\mathcal{A}(t,x)$ satisfies certain natural criteria): For all $\tilde{\tau} \le \tau$,

$$v(t,x) = \sup_{\alpha\in\mathcal{A}(t,x)} \mathbb{E}^{(t,x)}\Bigg[\int_t^{\tilde{\tau}} f(s,X^{\alpha}_s,\beta_s)ds + \sum_{\tau_j\le\tilde{\tau}} K(\tau_j,\check{X}^{\alpha}_{\tau_j-},\zeta_j) + v(\tilde{\tau},X^{\alpha}_{\tilde{\tau}})\Bigg]. \qquad (2.21)$$

The inequality $\leq$ follows from (2.20) and by approximation of the supremum. The inequality $\geq$ in (2.21) is obvious from the definition of the value function, basically if two admissible strategies applied sequentially in time form a new admissible strategy (see also Tang and Yong [111]). A similar DPP can be derived from (2.19); note however that neither this DPP nor (2.21) are used in our proofs.

2.4 Viscosity solution existence

In this section, we are going to prove the existence of an HJBQVI viscosity solution in the elliptic and parabolic case. Because a typical impulse control formulation will include the time, we will first prove the existence for the parabolic form, then transforming the problem including time component into a time-independent elliptic one (the problem formulation permitting).

2.4.1 Parabolic case

Recall the definition of $S_T := [0,T)\times S$ and its parabolic "boundary" $\partial^+ S_T := \big([0,T)\times(\mathbb{R}^d\setminus S)\big)\cup\big(\{T\}\times\mathbb{R}^d\big)$. We consider in this section the parabolic HJBQVI in the form

$$\begin{aligned}\min(-\sup_{\beta\in B}\{u_t+\mathcal{L}^\beta u+f^\beta\}, u-\mathcal{M}u) &= 0 \qquad \text{in } S_T\\ \min(u-g, u-\mathcal{M}u) &= 0 \qquad \text{in } \partial^+ S_T,\end{aligned} \tag{2.10}$$

for the integro-differential operator $\mathcal{L}^\beta$ from (2.11) (or infinitesimal generator of the process X), and the intervention operator $\mathcal{M}$ selecting the optimal instantaneous impulse.

Let us now define what exactly we mean by a viscosity solution of (2.10). Let $LSC(\Omega)$ (resp., $USC(\Omega)$) denote the set of measurable functions on the set Ω that are lower semicontinuous (resp., upper semicontinuous). Let $T>0$, and let u^* (u_*) define the upper (lower) semicontinuous envelope of a function u on $[0,T]\times\mathbb{R}^d$, i.e. the limit superior (limit inferior) is taken only from within this set. Let us also recall the definition of $\mathcal{PB}$ encapsulating the growth condition from §2.2.

Definition 2.4.1 (Viscosity solution). *A function $u\in\mathcal{PB}([0,T]\times\mathbb{R}^d)$ is a (viscosity) subsolution of (2.10) if for all $(t_0,x_0)\in[0,T]\times\mathbb{R}^d$ and $\varphi\in\mathcal{PB}\cap C^{1,2}([0,T)\times\mathbb{R}^d)$ with $\varphi(t_0,x_0)=u^*(t_0,x_0)$, $\varphi\geq u^*$ on $[0,T)\times\mathbb{R}^d$,*

$$\begin{aligned}\min\left(-\sup_{\beta\in B}\left\{\frac{\partial\varphi}{\partial t}+\mathcal{L}^\beta\varphi+f^\beta\right\}, u^*-\mathcal{M}u^*\right) &\leq 0 \qquad \textit{in } (t_0,x_0)\in S_T\\ \min\left(u^*-g, u^*-\mathcal{M}u^*\right) &\leq 0 \qquad \textit{in } (t_0,x_0)\in\partial^+ S_T.\end{aligned}$$

A function $u\in\mathcal{PB}([0,T]\times\mathbb{R}^d)$ is a (viscosity) supersolution of (2.10) if for all $(t_0,x_0)\in[0,T]\times\mathbb{R}^d$ and $\varphi\in\mathcal{PB}\cap C^{1,2}([0,T)\times\mathbb{R}^d)$ with $\varphi(t_0,x_0)=u_*(t_0,x_0)$, $\varphi\leq u_*$ on $[0,T)\times\mathbb{R}^d$,

$$\begin{aligned}\min\left(-\sup_{\beta\in B}\left\{\frac{\partial\varphi}{\partial t}+\mathcal{L}^\beta\varphi+f^\beta\right\}, u_*-\mathcal{M}u_*\right) &\geq 0 \qquad \textit{in } (t_0,x_0)\in S_T\\ \min\left(u_*-g, u_*-\mathcal{M}u_*\right) &\geq 0 \qquad \textit{in } (t_0,x_0)\in\partial^+ S_T.\end{aligned}$$

A function u is a viscosity solution if it is sub and supersolution.

The conditions on the parabolic boundary are included inside the viscosity solution definition (sometimes called "strong viscosity solution", see, e.g., Ishii [65]) because of the implicit form of this "boundary condition". In T, we chose the implicit form too, because otherwise the comparison result would not hold. The time derivative at $t = 0$ is of course to be understood as a one-sided derivative.

Now we can state the main result of the section, the existence theorem:

Theorem 2.4.2 (HJBQVI viscosity solution: existence)**.** *Let Assumptions 2.2.1 and 2.2.2 be satisfied. Then the value function v in* (2.7) *is a viscosity solution of (2.10) as defined above.*

For the proof of Theorem 2.4.2, we rely mainly on the proof given by Ly Vath et al. [80], extending it to a general setting with jumps; compare also the sketch of proof in Øksendal and Sulem [93]. We need a sequence of lemmas beforehand. The following lemma states first and foremost that the operator $\mathcal{M}$ preserves continuity. In a slightly different setting, the first two assertions can be found, e.g., in Lemma 5.5 of [80].

Lemma 2.4.3. *Let (V1), (V2) be satisfied for all parts except (v). Let u be a locally bounded function on $[0,T] \times \mathbb{R}^d$. Then:*

(i) $\mathcal{M}u_* \in LSC([0,T] \times \mathbb{R}^d)$ and $\mathcal{M}u_* \le (\mathcal{M}u)_*$.

(ii) $\mathcal{M}u^* \in USC([0,T] \times \mathbb{R}^d)$ and $(\mathcal{M}u)^* \le \mathcal{M}u^*$.

(iii) If $u \le \mathcal{M}u$ on $[0,T] \times \mathbb{R}^d$, then $u^* \le \mathcal{M}u^*$ on $[0,T] \times \mathbb{R}^d$.

(iv) For an approximating sequence $(t_n, x_n) \to (t,x)$, $(t_n, x_n) \subset [0,T] \times \mathbb{R}^d$ with $u(t_n, x_n) \to u^*(t,x)$: If $u^*(t,x) > \mathcal{M}u^*(t,x)$, then there exists $N \in \mathbb{N}$ such that $u(t_n,x_n) > \mathcal{M}u(t_n,x_n)$ $\forall n \ge N$.

(v) $\mathcal{M}$ is monotone, i.e. for $u \ge w$, $\mathcal{M}u \ge \mathcal{M}w$. In particular, for the value function v, $\mathcal{M}^n v \le v$ for all $n \ge 1$.

(vi) $\|\mathcal{M}u - \mathcal{M}w\|_{\infty,C} \le \|u - w\|_{\infty,\Gamma(C,Z(C))}$, where $\|\cdot\|_{\infty,C}$ is the supremum norm on some set $C \subset [0,T] \times \mathbb{R}^d$.

Proof: (i): Let $(t_n, x_n)_n$ be a sequence in $[0,T] \times \mathbb{R}^d$ converging to (t,x). For an $\varepsilon > 0$, select $\zeta^\varepsilon \in Z(t,x)$ with $u_*(t, \Gamma(t,x,\zeta^\varepsilon)) + K(t,x,\zeta^\varepsilon) + \varepsilon \ge \mathcal{M}u_*(t,x)$. Choose a sequence $\zeta_n \to \zeta^\varepsilon$ with $\zeta_n \in Z(t_n,x_n)$ for all n (possible by Hausdorff convergence). Then

$$\begin{aligned}\liminf_{n\to\infty} \mathcal{M}u_*(t_n,x_n) &\ge \liminf_{n\to\infty} u_*(t, \Gamma(t_n,x_n,\zeta_n)) + K(t_n,x_n,\zeta_n) \\ &\ge u_*(t, \Gamma(t,x,\zeta^\varepsilon)) + K(t,x,\zeta^\varepsilon) \ge \mathcal{M}u_*(t,x) - \varepsilon.\end{aligned}$$

The second assertion follows because $\mathcal{M}u \ge \mathcal{M}u_*$, and thus $(\mathcal{M}u)_* \ge (\mathcal{M}u_*)_* = \mathcal{M}u_*$.

(ii): Fix some $(t,x) \in [0,T] \times \mathbb{R}^d$, and let $(t_n,x_n) \in [0,T] \times \mathbb{R}^d$ converge to (t,x). Because of the upper semicontinuity of u^* and continuity of Γ and K, for each fixed n, the maximum in $\mathcal{M}u^*(t_n,x_n)$ is achieved, i.e., there is $\zeta_n \in Z(t_n,x_n)$ such that $\mathcal{M}u^*(t_n,x_n) = u^*(t_n, \Gamma(t_n,x_n,\zeta_n)) + K(t_n,x_n,\zeta_n)$. $(\zeta_n)_n$ is contained in a bounded set, thus has a convergent subsequence with limit $\hat{\zeta} \in Z(t,x)$ (assume that $\hat{\zeta} \notin Z(t,x)$, then $dist(\hat{\zeta}, Z(t,x)) > 0$, which contradicts the Hausdorff

convergence).
We get

$$\begin{aligned}\mathcal{M}u^*(t,x) &\geq u^*(t,\Gamma(t,x,\hat{\zeta})) + K(t,x,\hat{\zeta}) \geq \limsup_{n\to\infty} u^*(t_n,\Gamma(t_n,x_n,\zeta_n)) + K(t_n,x_n,\zeta_n) \\ &= \limsup_{n\to\infty} \mathcal{M}u^*(t_n,x_n).\end{aligned}$$

The second assertion follows because $\mathcal{M}u \leq \mathcal{M}u^*$, and thus $(\mathcal{M}u)^* \leq (\mathcal{M}u^*)^* = \mathcal{M}u^*$.

(iii): Follows immediately by (ii): If $u \leq \mathcal{M}u$, then $u^* \leq (\mathcal{M}u)^* \leq \mathcal{M}u^*$.

(iv): By contradiction: Assume $u(t_n,x_n) \leq \mathcal{M}u(t_n,x_n)$ for infinitely many n. Then by convergence along a subsequence,

$$u^*(t,x) \leq \limsup_{n\to\infty} \mathcal{M}u(t_n,x_n) \leq (\mathcal{M}u)^*(t,x) \leq \mathcal{M}u^*(t,x).$$

(v): The monotonicity follows directly from the definition of $\mathcal{M}$. $\mathcal{M}v \leq v$ is necessary for the value function v, because v is already optimal, and $\mathcal{M}^n v \leq v$ for all $n \geq 1$ then follows by induction.

(vi): Follows from $\sup_\zeta a(\zeta) - \sup_\zeta b(\zeta) \leq \sup_\zeta\{a(\zeta) - b(\zeta)\}$. □

By (V1), (V2) of §2.2, we obtain that $\mathcal{M}v(t,x) < \infty$ if v locally bounded. This finiteness and the property that there is a convergent subsequence of (ζ_n) are sufficient for (ii) (at least after reformulating the proof).

The existence proof frequently makes use of stopping times to ensure that a stochastic process X (started at x) is contained in some (small) set. This works very well for continuous processes, because then for a stopping time $\tau = \inf\{t \geq 0 : |X_t - x| \geq \rho_1\} \wedge \rho_2$, the process $|X_\tau - x| \leq \rho_1$. For a process including (non-predictable) jumps however, $|X_\tau - x|$ may be $> \rho_1$. Luckily, Lévy processes are stochastically continuous, which means that at least the *probability* of X_τ being outside $\overline{B(x,\rho_1)}$ converges to 1, if $\rho_2 \to 0$. Stochastic continuity as well holds for normal right-continuous Markov processes (see Dynkin [39], Lemma 3.2), and thus for our SDE solutions.

The lemma (Lemma A.2.2 in the appendix) destined to overcome this problem essentially states the fact that stochastically continuous processes on a compact time interval are uniformly stochastically continuous. A further lemma (Lemma A.2.1 in the appendix) shows that for a continuously controlled process, stochastic continuity holds true uniformly in the control (this is of course a consequence of (E4)).

Now we are ready for the proof of the existence theorem. Recall that a necessary condition for the value function on the parabolic boundary is

$$\min(v - g, v - \mathcal{M}v) = 0 \qquad \text{in } \partial^+ S_T. \tag{2.12}$$

Proof of Theorem 2.4.2: **v is supersolution:** First, for any $(t_0,x_0) \in [0,T] \times \mathbb{R}^d$, the inequality $v(t_0,x_0) \geq \mathcal{M}v(t_0,x_0)$ holds, because otherwise an immediate jump would increase the value function. By Lemma 2.4.3 (i), $\mathcal{M}v_*(t_0,x_0) \leq (\mathcal{M}v)_*(t_0,x_0) \leq v_*(t_0,x_0)$.

We then verify the condition on the parabolic boundary: Since we can decide to stop immediately, $v \geq g$ on $\partial^+ S_T$, so $v_* \geq g$ follows by the continuity of g (outside $\overline{S}$) and requirement (E3) (if $x_0 \in \partial S$ or $t_0 = T$).

So it remains to show the other part of the inequality

$$-\sup_{\beta\in B}\left\{\frac{\partial\varphi}{\partial t}+\mathcal{L}^{\beta}\varphi+f^{\beta}\right\}\geq 0 \tag{2.22}$$

in a fixed $(t_0,x_0)\in[0,T)\times S$, for $\varphi\in\mathcal{PB}\cap C^{1,2}([0,T)\times\mathbb{R}^d)$, $\varphi(t_0,x_0)=v_*(t_0,x_0)$, $\varphi\leq v_*$ on $[0,T)\times\mathbb{R}^d$.
From the definition of v_*, there exists a sequence $(t_n,x_n)\in[0,T)\times S$ such that $(t_n,x_n)\to(t_0,x_0)$, $v(t_n,x_n)\to v_*(t_0,x_0)$ for $n\to\infty$. By continuity of φ, $\delta_n:=v(t_n,x_n)-\varphi(t_n,x_n)$ converges from above to 0 as n goes to infinity. Because $(t_0,x_0)\in[0,T)\times S$, there exists $\rho>0$ such that for n large enough, $t_n<T$ and $B(x_n,\rho)\subset B(x_0,2\rho)=\{|y-x_0|<2\rho\}\subset S$.
Let us now consider the combined control with no impulses and a constant stochastic control $\beta\in B$, and the corresponding controlled stochastic process X^{β,t_n,x_n} starting in (t_n,x_n). Choose a strictly positive sequence (h_n) such that $h_n\to 0$ and $\delta_n/h_n\to 0$ as $n\to\infty$. For

$$\bar\tau_n:=\inf\{s\geq t_n:|X^{\beta,t_n,x_n}-x_n|\geq\rho\}\wedge(t_n+h_n)\wedge T,$$

we get by the strong Markov property and Dynkin's formula for ρ sufficiently small ($\mathbb{E}^n=\mathbb{E}^{(t_n,x_n)}$ denotes the expectation when X starts in t_n with x_n):

$$\begin{aligned}
v(t_n,x_n) &\geq \mathbb{E}^n\left[\int_{t_n}^{\bar\tau_n} f(s,X_s^{\beta},\beta)ds+v(\bar\tau_n,\check X^{\beta}_{\bar\tau_n-})\right]\\
&\geq \mathbb{E}^n\left[\int_{t_n}^{\bar\tau_n} f(s,X_s^{\beta},\beta)ds+\varphi(\bar\tau_n,\check X^{\beta}_{\bar\tau_n-})\right]\\
&= \varphi(t_n,x_n)+\mathbb{E}^n\left[\int_{t_n}^{\bar\tau_n} f(s,X_s^{\beta},\beta)+\frac{\partial\varphi}{\partial t}(s,X_s^{\beta})+\mathcal{L}^{\beta}\varphi(s,X_s^{\beta})\,ds\right]
\end{aligned}$$

Here, our assumptions on the SDE coefficients of X were sufficient to apply Dynkin's formula because of the localizing stopping time $\bar\tau_n$ (compare §A.1). Using the definition of δ_n, we obtain

$$\frac{\delta_n}{h_n}\geq\mathbb{E}^n\left[\frac{1}{h_n}\int_{t_n}^{\bar\tau_n} f(s,X_s^{\beta},\beta)+\frac{\partial\varphi}{\partial t}(s,X_s^{\beta})+\mathcal{L}^{\beta}\varphi(s,X_s^{\beta})\,ds\right]. \tag{2.23}$$

Now, we want to let converge $n\to\infty$ in (2.23), but it is not possible to apply the mean value theorem because $s\mapsto f(s,X_s^{\beta},\beta)$ (for fixed ω) is in general not continuous. Select $\varepsilon\in(0,\rho)$. By Lemma A.2.2, $\mathbb{P}(\sup_{t_n\leq s\leq r}|X_s^{\beta,t_n,x_n}-x_n|>\varepsilon)\to 0$ for $r\downarrow t_n$. Define now

$$A_{n,\varepsilon}=\{\omega:\sup_{t_n\leq s\leq t_n+h_n}|X_s^{\beta,t_n,x_n}-x_n|\leq\varepsilon\},$$

and split the integral in (2.23) into two parts

$$\left(\int_{t_n}^{t_n+h_n}\right)1_{A_{n,\varepsilon}}+\left(\int_{t_n}^{\bar\tau_n}\right)1_{A^c_{n,\varepsilon}}.$$

On $A_{n,\varepsilon}$, for the integrand G of the right hand side in (2.23),

$$\begin{aligned}
\left|G(t_0,x_0,\beta)-\frac{1}{h_n}\int_{t_n}^{t_n+h_n}G(s,X_s^{\beta},\beta)\,ds\right| &\leq \frac{1}{h_n}\int_{t_n}^{t_n+h_n}|G(t_0,x_0,\beta)-G(s,X_s^{\beta},\beta)|\,ds\\
&\leq |G(t_0,x_0,\beta)-G(\hat t_{n,\varepsilon},\hat x_{n,\varepsilon},\beta)|, \qquad (2.24)
\end{aligned}$$

the latter because G is continuous by (V3) and assumption on φ, and the maximum distance of $|G(t_0, x_0, \beta) - G(\cdot, \cdot, \beta)|$ is assumed in a $(\hat{t}_{n,\varepsilon}, \hat{x}_{n,\varepsilon}) \in [t_n, t_n + h_n] \times \overline{B(x_n, \varepsilon)}$.
On the complement of $A_{n,\varepsilon}$,

$$\frac{1}{h_n}\left(\int_{t_n}^{\bar{\tau}_n}\right) 1_{A^c_{n,\varepsilon}} \leq \operatorname{ess}\sup_{t_n \leq s \leq \bar{\tau}_n} \left| f(s, X_s^\beta, \beta) + \frac{\partial \varphi}{\partial t}(s, X_s^\beta) + \mathcal{L}^\beta \varphi(s, X_s^\beta) \right| 1_{A^c_{n,\varepsilon}},$$

which is bounded by the same arguments as above and because a jump in $\bar{\tau}_n$ does not affect the essential supremum.

Because $h_n \to 0$ and $(t_n, x_n) \to (t_0, x_0)$ for $n \to \infty$ and by stochastic continuity, $\mathbb{P}(A_{n,\varepsilon}) \to 1$, for all $\varepsilon > 0$ or, equivalently, $1_{A_{n,\varepsilon}} \to 1$ almost surely. So by $n \to \infty$ and then $\varepsilon \to 0$, we can conclude by the dominated convergence theorem that $f(t_0, x_0, \beta) + \frac{\partial \varphi}{\partial t}(t_0, x_0) + \mathcal{L}^\beta \varphi(t_0, x_0) \leq 0$ $\forall \beta \in B$, and thus (2.22) holds. □

v **is subsolution:** Let $(t_0, x_0) \in [0, T] \times \mathbb{R}^d$ and $\varphi \in \mathcal{PB} \cap C^{1,2}([0, T) \times \mathbb{R}^d)$ such that $v^*(t_0, x_0) = \varphi(t_0, x_0)$ and $\varphi \geq v^*$ on $[0, T) \times \mathbb{R}^d$. If $v^*(t_0, x_0) \leq \mathcal{M}v^*(t_0, x_0)$, then the subsolution inequality holds trivially. So consider from now on the case $v^*(t_0, x_0) > \mathcal{M}v^*(t_0, x_0)$.

Consider $(t_0, x_0) \in \partial^+ S_T$. For an approximating sequence $(t_n, x_n) \to (t_0, x_0)$ in $[0, T] \times \mathbb{R}^d$ with $v(t_n, x_n) \to v^*(t_0, x_0)$, the relation $v(t_n, x_n) > \mathcal{M}v(t_n, x_n)$ holds by Lemma 2.4.3 (iv). Thus by the continuity of g (outside $\overline{S}$) and requirement (E3) (if $x_0 \in \partial S$ or $t_0 = T$),

$$g(t_0, x_0) = \lim_{n \to \infty} g(t_n, x_n) = \lim_{n \to \infty} v(t_n, x_n) = v^*(t_0, x_0).$$

Now let us show that, for $v^*(t_0, x_0) > \mathcal{M}v^*(t_0, x_0)$,

$$-\sup_{\beta \in B} \left\{ \frac{\partial \varphi}{\partial t} + \mathcal{L}^\beta \varphi + f^\beta \right\} \leq 0 \tag{2.25}$$

in $(t_0, x_0) \in [0, T) \times S$. We argue by contradiction and assume that there is an $\eta > 0$ such that

$$\sup_{\beta \in B} \left\{ \frac{\partial \varphi}{\partial t} + \mathcal{L}^\beta \varphi + f^\beta \right\} < -\eta < 0. \tag{2.26}$$

Because $\varphi \in \mathcal{PB} \cap C^{1,2}([0, T) \times \mathbb{R}^d)$ and the Hamiltonian is continuous in (t, x) by (V3), there is an open set $\mathcal{O}$ surrounding (t_0, x_0) in S_T where $\sup_{\beta \in B} \left\{ \frac{\partial \varphi}{\partial t} + \mathcal{L}^\beta \varphi + f^\beta \right\} < -\eta/2$.

From the definition of v^*, there exists a sequence $(t_n, x_n) \in S_T \cap \mathcal{O}$ such that $(t_n, x_n) \to (t_0, x_0)$, $v(t_n, x_n) \to v^*(t_0, x_0)$ for $n \to \infty$. By continuity of φ, again $\delta_n := v(t_n, x_n) - \varphi(t_n, x_n)$ converges to 0 as n goes to infinity.

By definition of the value function, there exists for all n and $\varepsilon > 0$ (choose $\varepsilon = \varepsilon_n$ with $\varepsilon_n \downarrow 0$) a combined admissible control $\alpha^n = \alpha^n(\varepsilon) = (\beta^n, \gamma^n)$, $\gamma^n = (\tau_i^n, \zeta_i^n)_{i \geq 1}$ such that

$$v(t_n, x_n) \leq J^{(\alpha^n)}(t_n, x_n) + \varepsilon. \tag{2.27}$$

For a $\rho > 0$ chosen suitably small (i.e. $B(x_n, \rho) \subset B(x_0, 2\rho) \subset S$, $t_n + \rho < t_0 + 2\rho < T$ for large n), we define the stopping time $\bar{\tau}_n := \bar{\tau}_n^\rho \wedge \tau_1^n$, where

$$\bar{\tau}_n^\rho := \inf\{s \geq t_n : |X_s^{\alpha^n, t_n, x_n} - x_n| \geq \rho\} \wedge (t_n + \rho).$$

We want to show that $\bar{\tau}_n \to t_0$ in probability. From (2.27) combined with the Markov property (2.19), it immediately follows that (again $\mathbb{E}^n = \mathbb{E}^{(t_n,x_n)}$, and $\beta = \beta^n$, $\alpha = \alpha^n$)[5]

$$\begin{aligned} v(t_n, x_n) &\leq \mathbb{E}^n \left[\int_{t_n}^{\bar{\tau}_n} f(s, X_s^\beta, \beta_s) ds + v(\bar{\tau}_n, \check{X}^\beta_{\bar{\tau}_n -}) \right] + \varepsilon_n \\ &\leq \mathbb{E}^n \left[\int_{t_n}^{\bar{\tau}_n} f(s, X_s^\beta, \beta_s) ds + \varphi(\bar{\tau}_n, \check{X}^\beta_{\bar{\tau}_n -}) \right] + \varepsilon_n. \end{aligned} \tag{2.28}$$

Thus by Dynkin's formula on $\varphi(\bar{\tau}_n, \check{X}^\beta_{\bar{\tau}_n -})$ and using $v(t_n, x_n) = \varphi(t_n, x_n) + \delta_n$,

$$\begin{aligned} \delta_n &\leq \mathbb{E}^n \left[\int_{t_n}^{\bar{\tau}_n} f(s, X_s^\beta, \beta_s) + \frac{\partial \varphi}{\partial s}(s, X_s^\beta) + \mathcal{L}^\beta \varphi(s, X_s^\beta) ds \right] + \varepsilon_n \\ &\leq -\frac{\eta}{2} \mathbb{E}^n [\bar{\tau}_n - t_n] + \varepsilon_n, \end{aligned}$$

where for ρ small enough, we have applied (2.26). This implies that $\lim_{n\to\infty} \mathbb{E}[\bar{\tau}_n] = t_0$, which is equivalent to $\bar{\tau}_n \to t_0$ in probability (as one can easily check with Chebyshev's inequality; $\bar{\tau}_n$ is bounded).

Now let us continue with our proof. In the following, we make again use of the stochastic continuity of X^{β_n, t_n, x_n} (up until the first impulse). We define

$$A_n(\rho) = \{\omega : \sup_{t_n \leq s \leq \bar{\tau}_n} |X_s^{\beta_n, t_n, x_n} - x_n| \leq \rho\}.$$

(2.27) combined with the Markov property (2.20) gives us

$$v(t_n, x_n) \leq \mathbb{E}^n \left[\int_{t_n}^{\bar{\tau}_n} f(s, X_s^\beta, \beta_s) ds + K(\tau_1^n, \check{X}^\beta_{\tau_1^n -}, \zeta_1^m) 1_{\bar{\tau}_n^\rho \geq \tau_1^n} + v(\bar{\tau}_n, X^\alpha_{\bar{\tau}_n}) \right] + \varepsilon_n. \tag{2.29}$$

To find upper estimates for $v(t_n, x_n)$, we use indicator functions for three separate cases:

$$\{\bar{\tau}_n^\rho < \tau_1^n\} \tag{I}$$
$$\{\bar{\tau}_n^\rho \geq \tau_1^n\} \cap A_n(\rho)^c \tag{II}$$
$$\{\bar{\tau}_n^\rho \geq \tau_1^n\} \cap A_n(\rho) \tag{III}$$

(III) is the predominant set: For any sequence $(\hat{\varepsilon}_n)$, by basic probability $\mathbb{P}(\bar{\tau}_n^\rho \geq \tau_1^n) \geq 1 - \mathbb{P}(\bar{\tau}_n^\rho < \hat{\varepsilon}_n) - \mathbb{P}(\tau_1^n \geq \hat{\varepsilon}_n)$. Choose $\hat{\varepsilon}_n \downarrow t_0$ such that $\mathbb{P}(\tau_1^n \geq \hat{\varepsilon}_n) \to 0$. By Lemma A.2.2 (iii) (wlog, we need only consider the setting without impulses), also $\limsup_{n\to\infty} \mathbb{P}(\bar{\tau}_n^\rho < \hat{\varepsilon}_n) = 0$. In total, we have $\mathbb{P}(III) \to 1$ or $1_{(III)} \to 1$ a.s. for $n \to \infty$.

Thus, using that if there is an impulse in $\bar{\tau}_n$ (i.e. $\bar{\tau}_n^\rho \geq \tau_1^n$), then $v(\bar{\tau}_n, X^\alpha_{\bar{\tau}_n}) + K(\bar{\tau}_n, \check{X}^\beta_{\bar{\tau}_n -}, \zeta_1^n) \leq \mathcal{M} v(\bar{\tau}_n, \check{X}^\beta_{\bar{\tau}_n -})$,[6]

$$\begin{aligned} v(t_n, x_n) \leq & \sup_{\substack{|t'-t_0|<\rho \\ |y'-x_0|<\rho}} f(t', y', \beta_{t'}) \mathbb{E}[\bar{\tau}_n - t_n] \\ & + \mathbb{E}[|v(\bar{\tau}_n, X^\beta_{\bar{\tau}_n})| 1_{(I)}] + \mathbb{E}[|\mathcal{M} v(\bar{\tau}_n, \check{X}^\beta_{\bar{\tau}_n -})| 1_{(II)}] \\ & + \sup_{\substack{|t'-t_0|<\rho \\ |y'-x_0|<\rho}} \mathcal{M} v(t', y') \mathbb{E} 1_{(III)} + \varepsilon_n. \end{aligned}$$

[5] In the following, we will switch between α and β in our notation, where the usage of β indicates that there is no impulse to take into account.

[6] Note: More than one impulse could occur in $\bar{\tau}_n$ if the transaction cost structure allows for it (e.g., K quadratic in ζ). In this case however, the result follows by monotonicity of $\mathcal{M}$ (Lemma 2.4.3 (v)).

To prove the boundedness in term (I) (uniform in n), we can assume wlog that $\nu(\mathbb{R}^k) = 1$, and consider only jumps bounded away from 0 for $x_0 > 0$. Then by (E1), $\mathbb{E}\sup_n |v(\bar\tau_n, X^\beta_{\bar\tau_n})| \leq C\,\mathbb{E}[1 + R(x_0 + 2\rho + Y)]$ for a jump Y with distribution $\nu^{\ell(t_0+2\rho,x_0+2\rho,\beta,\cdot)}$, which is finite by the definition of $\mathcal{PB}$. The same is true for $\mathcal{M}v(\bar\tau_n, \check X^\beta_{\bar\tau_n -})$ (for $\mathcal{M}v \geq 0$ because $\mathcal{M}v \leq v$, for negative $\mathcal{M}v$ this follows from the definition).

Sending $n \to \infty$ ($\limsup_{n\to\infty}$), the f-term, and term (I) converge to 0 by the dominated convergence theorem. For term (II), a general version of the DCT shows that it is bounded by

$$\mathbb{E}[\limsup_{n\to\infty} |\mathcal{M}v(\bar\tau_n, \check X^\beta_{\bar\tau_n -})| 1_{A_n(\rho)^c}],$$

and term (III) becomes $\sup \mathcal{M}v(t', y')$. Now we let $\rho \to 0$ and obtain:

$$v^*(t_0, x_0) \leq \lim_{\rho\downarrow 0} \sup_{\substack{|t'-t_0|<\rho \\ |y'-x_0|<\rho}} \mathcal{M}v(t', y') = (\mathcal{M}v)^*(t_0, x_0) \leq \mathcal{M}v^*(t_0, x_0)$$

by Lemma 2.4.3 (ii), a contradiction. Thus (2.25) is true. □

Let us elaborate on some details of the proof:

- In the proof, we have only used that all (piecewise) constant stochastic controls with values in B are admissible for the SDE (2.4). So actually, we are quite free how to choose the set of admissible controls — the value function always turns out to be a viscosity solution.
- Another approach for the subsolution part would be tempting, although we do not see how this can work: In the subsolution proof, we assumed $v^*(t_0, x_0) - \mathcal{M}v^*(t_0, x_0) > 0$. This implies, using Lemma 2.4.3 (iv) and the 0-1 law, that for n large enough, $\tau_1^n > t_n$ a.s. On the other hand, from $\bar\tau_n \to t_0$, it follows by Lemma A.2.1 that $\tau_1^n \to t_0$ in probability (it is sufficient to consider the setting of Lemma A.2.1 without impulse, since otherwise the first impulse would anyhow converge to 0 in probability). So the convergence of τ_1^n points already to a contradiction.
- By local boundedness of v, the derivation of the dynamic programming principle and the integrability condition (2.8), we could already deduce that $\mathbb{E}^{t_n,x_n}[v(\bar\tau_n, X^\beta_{\bar\tau_n})] < \infty$; it is however not so easy to deduce this uniformly in n. The condition (2.8) contains implicitly conditions on ν we have formalized in (V1).

We promised to come back to the "regularity at ∂S" issue, and present here conditions sufficient for condition (E3).

(E1*) For any point $(t, x) \in [0, T) \times \partial S$, any sequence $(t_n, x_n) \subset [0, T) \times S$, $(t_n, x_n) \to (t, x)$, and for each small $\varepsilon > 0$, there is an admissible combined control $\alpha_n = (\beta_n, \gamma_n)$ such that

$$v(t_n, x_n) \leq J^{(\alpha_n)}(t_n, x_n) + \varepsilon, \tag{2.30}$$

and such that for all $\delta > 0$, $\mathbb{P}(\tilde\tau_S^n < t_n + \delta) \to 1$ for $n \to \infty$ (where $\tilde\tau_S^n = \inf\{s \geq t_n : X^{\beta_n,t_n,x_n}(s) \notin S\}$).

(E2*) For any point $(t, x) \in \partial^* S_T$, if there is a sequence $(t_n, x_n) \subset [0, T) \times S$ converging to (t, x) with $v(t_n, x_n) \to v^*(t, x)$ and $v^*(t, x) > \mathcal{M}v^*(t, x)$, then there is a neighbourhood of (t, x) (open in $[0, T] \times \mathbb{R}^d$), where $v > \mathcal{M}v$.

Example 2.4.1. Let X be a one-dimensional Brownian motion with $\sigma > 0$, and assume it is never optimal to give an impulse near the boundary. Then (E1*) and (E2*) are satisfied.

We define $\tau_S^n = \inf\{s \geq t_n : X^{\alpha_n, t_n, x_n}(s) \notin S\}$.

Proposition 2.4.4. *Let $\tau = \tau_S$ or $\tau = \tau_S \wedge T$. If (E1*) and (E2*) hold, and for n large the integrability condition*

$$\int_t^\tau |f(s, X_s^{\alpha_n}, (\beta_n)_s)|ds + |g(\tau, X_\tau^{\alpha_n})|1_{\tau<\infty} + \sum_{\tau_j \leq \tau} |K(\tau_j^n, \check{X}^{\alpha_n}_{\tau_j^n-}, \zeta_j^n)| \leq h \in L^1(\mathbb{P}; \mathbb{R})$$

is satisfied, then (E3) holds.

Proof: Let $(t, x) \in [0, T) \times \partial S$ and assume wlog that $v > \mathcal{M}v$ in a neighbourhood of (t, x). Then we have to show for all $\varepsilon > 0$ and all sequences chosen as in (E1*), that

$$\mathbb{E}^{(t_n,x_n)}\left[\int_{t_n}^{\tau_S} f(s, X_s^{\alpha_n}, (\beta_n)_s)ds + g(\tau_S, X_{\tau_S}^{\alpha_n})1_{\tau_S<\infty} + \sum_{\tau_j \leq \tau_S} K(\tau_j^n, \check{X}^{\alpha_n}_{\tau_j^n-}, \zeta_j^n)\right] \to g(t, x) \tag{2.31}$$

as $n \to \infty$, $\delta \to 0$. For the set

$$B_{n,\delta} := \{\tau_S^n < t_n + \delta\} \cap \{\sup_{t_n \leq t \leq \tau_S^n} |X_t^{\alpha_n, t_n, x_n} - x_n| < \delta\},$$

we claim that for all $\delta > 0$ small enough, $1_{B_{n,\delta}} \to 1$ as $n \to \infty$. To see this, first note that by assumption and Lemma A.2.2 (iii), this is true for the set

$$\tilde{B}_{n,\delta} := \{\tilde{\tau}_S^n < t_n + \delta\} \cap \{\sup_{t_n \leq t \leq \tilde{\tau}_S^n} |X_t^{\beta_n, t_n, x_n} - x_n| < \delta\}.$$

Choose δ small enough such that $v > \mathcal{M}v$ on $B(x_n, \delta)$ and $x \in B(x_n, \delta)$ for all n large. Then it is easily checked that $\{\sup_{t_n \leq t \leq \tau_S^n} |X_t^{\alpha_n, t_n, x_n} - x_n| < \delta\} = \{\sup_{t_n \leq t \leq \tilde{\tau}_S^n} |X_t^{\beta_n, t_n, x_n} - x_n| < \delta\}$. Because

$$\mathbb{P}\left(\tau_S^n < t_n + \delta \,\middle|\, \sup_{t_n \leq t \leq \tau_S^n} |X_t^{\beta_n, t_n, x_n} - x_n| < \delta\right) = \mathbb{P}\left(\tilde{\tau}_S^n < t_n + \delta \,\middle|\, \sup_{t_n \leq t \leq \tilde{\tau}_S^n} |X_t^{\beta_n, t_n, x_n} - x_n| < \delta\right)$$

and the latter expression converges to 1 for $n \to \infty$, we can conclude that $1_{B_{n,\delta}} \to 1$ as $n \to \infty$ for $\delta > 0$ small enough.

The convergence in (2.31) then follows just as in the existence proof, by splitting into $B_{n,\delta}$ and $B_{n,\delta}^c$ and applying the lim sup, lim inf versions of the dominated convergence theorem, first for $n \to \infty$, and then for $\delta \to 0$ (using the continuity of f, g). The result for $t = T$ holds by the same arguments, because time is always regular. $\square$

Remark 2.4.1. (E3) (resp, (E1*), (E2*)) excludes in particular problems with *de facto* state constraints, where it is optimal to stay inside S. We note however that the framework presented here allows for an adaptation to (true and *de facto*) state constraints, which can be pretty straightforward for easy constraints. Apart from the stochastic proof that we can restrain ourselves to controls keeping the process inside S, the adaptation involves changing the function w used in the uniqueness part, such that only values in S need to be considered in the comparison proof. For an example in the diffusion case, see Ly Vath et al. [80]; jumps outside S however may be difficult to handle.

2.4.2 Elliptic case

The existence result for the elliptic QVI (2.15) now follows from the parabolic result by an exponential time transformation. Recall that the elliptic QVI is

$$\min(-\sup_{\beta\in B}\{-\rho u+\mathcal{L}^\beta u+f^\beta\}, u-\mathcal{M}u)=0, \tag{2.15}$$

for the elliptic integro-differential operator $\mathcal{L}^\beta$ from (2.16) and the intervention operator $\mathcal{M}$ selecting the optimal instantaneous impulse.

For an $s\geq 0$ and $\rho>0$, let the functions as used in §2.4.1 be tagged by a tilde. $\tilde{f}$, $\tilde{g}$, $\tilde{K}$ are all built in the same way on $[0,\infty)\times\mathbb{R}^d$, as the following example:

$$\tilde{f}(t,x,\beta)=e^{-\rho(s+t)}f(x,\beta)$$

Let $\tilde{\Gamma}=\Gamma$ (independent of t) and likewise $\tilde{\mu},\tilde{\sigma},\tilde{\ell}$ and the transaction set $\tilde{Z}$. The intervention operator including time (as in §2.4.1) is denoted by $\tilde{\mathcal{M}}$, and the time-independent one is defined by

$$\mathcal{M}u(x)=\sup\{u(\Gamma(x,\zeta))+K(x,\zeta):\zeta\in Z(x)\}.$$

It can be checked that the assumptions of §2.4.1 hold for the tilde functions, if the corresponding assumption holds for the time-independent functions without tilde. As well, all the lemmas used for the proof of the existence theorem are still valid in the time-independent case.

Definition 2.4.5 (Viscosity solution)**.** *A function $u\in\mathcal{PB}(\mathbb{R}^d)$ is a (viscosity) subsolution of (2.15) if for all $x_0\in\mathbb{R}^d$ and $\varphi\in\mathcal{PB}\cap C^2(\mathbb{R}^d)$ with $\varphi(x_0)=u^*(x_0)$, $\varphi\geq u^*$,*

$$\begin{aligned}\min\left(-\sup_{\beta\in B}\left\{-\rho\varphi+\mathcal{L}^\beta\varphi+f^\beta\right\},u^*-\mathcal{M}u^*\right) &\leq 0 &&\textit{in } x_0\in S\\ \min\left(u^*-g,u^*-\mathcal{M}u^*\right) &\leq 0 &&\textit{in } x_0\in\mathbb{R}^d\setminus S.\end{aligned}$$

A function $u\in\mathcal{PB}(\mathbb{R}^d)$ is a (viscosity) supersolution of (2.15) if for all $x_0\in\mathbb{R}^d$ and $\varphi\in\mathcal{PB}\cap C^2(\mathbb{R}^d)$ with $\varphi(x_0)=u_*(x_0)$, $\varphi\leq u_*$,

$$\begin{aligned}\min\left(-\sup_{\beta\in B}\left\{-\rho\varphi+\mathcal{L}^\beta\varphi+f^\beta\right\},u_*-\mathcal{M}u_*\right) &\geq 0 &&\textit{in } x_0\in S\\ \min\left(u_*-g,u_*-\mathcal{M}u_*\right) &\geq 0 &&\textit{in } x_0\in\mathbb{R}^d\setminus S.\end{aligned}$$

A function u is a viscosity solution if it is sub and supersolution.

In the original problem on $[0,\infty)\times\mathbb{R}^d$, we only consider Markov controls that are time-independent, i.e., only depend on the state variable x. Denote by $\tilde{v}$ the then resulting value function of the parabolic problem on $[0,\infty)\times\mathbb{R}^d$. Then we can define the time-independent value function

$$v(x):=e^{\rho(s+t)}\tilde{v}(t,x). \tag{2.32}$$

Let us emphasize that this definition is only admissible if the right-hand side actually does not depend on t, which can be checked in the definition of $J^{(\alpha)}(t,x)$ by the homogeneous Markov property (and of course only if the time horizon is infinite, i.e., $\tau=\tau_S$ for $S\subset\mathbb{R}^d$).

The existence for the elliptic QVI then follows by an easy time transformation:

Corollary 2.4.6. *Let Assumptions 2.2.1 and 2.2.2 be satisfied. Then the value function v as defined above is a viscosity solution of (2.15).*

Proof: We know from Theorem 2.4.2 that $\tilde{v}$ is a (parabolic) viscosity solution of (2.10) on $[0,\infty) \times \mathbb{R}^d$ (without the terminal condition), i.e. of

$$\begin{aligned} \min(-\sup_{\beta \in B}\{u_t + \mathcal{L}^\beta u + \tilde{f}^\beta\}, u - \tilde{\mathcal{M}}u) = 0 \qquad & \text{in } (t_0, x_0) \in [0,\infty) \times S \\ \min(u - \tilde{g}, u - \tilde{\mathcal{M}}u) = 0 \qquad & \text{in } (t_0, x_0) \in [0,\infty) \times (\mathbb{R}^d \setminus S). \end{aligned} \tag{2.33}$$

For the subsolution proof, let $\varphi \in \mathcal{PB} \cap C^2(\mathbb{R}^d)$ with $\varphi(x_0) = v^*(x_0)$, $\varphi \geq v^*$. For $\tilde{\varphi}(t,x) := e^{-\rho(s+t)}\varphi(x)$ for all $t \geq 0$, $\tilde{\varphi}(t_0, x_0) = \tilde{v}^*(t_0, x_0)$ and $\tilde{\varphi} \geq \tilde{v}^*$. Furthermore,

$$\tilde{\varphi}_t + \mathcal{L}^\beta \tilde{\varphi} = e^{-\rho(s+t)}(-\rho\varphi + \mathcal{L}^\beta \varphi).$$

By the definition of the elliptic $\mathcal{M}$, we have $\tilde{v} - \tilde{\mathcal{M}}\tilde{v} = e^{-\rho(s+t)}(v - \mathcal{M}v)$. The supersolution property is proved in the same manner. □

2.5 Viscosity solution uniqueness

The purpose of this section is to prove uniqueness results both for the elliptic and the parabolic HJBQVI by analytic means. Using such a uniqueness result, together with the existence results of §2.4, we can conclude

The viscosity solution of the HJBQVI is equal to the value function.

We were inspired mainly by the papers Ishii [65] (for the impulse part) and Barles and Imbert [11] (for the PIDE part). As general reference for viscosity solutions, Crandall et al. [33] was used and will be frequently cited. Some ideas have come from [92], [93], [3] and [68].

In this section, v does not denote the value function any longer, and some other variables may serve new purposes as well.

First, we will investigate uniqueness of HJBQVI viscosity solutions for the elliptic case (the parabolic case following at the end):

$$\begin{aligned} \min(-\sup_{\beta \in B}\{-cu + \mathcal{L}^\beta u + f^\beta\}, u - \mathcal{M}u) = 0 \qquad & \text{in } S \\ \min(u - g, u - \mathcal{M}u) = 0 \qquad & \text{in } \mathbb{R}^d \setminus S, \end{aligned} \tag{2.34}$$

where c is some positive function related to the discounting in the original model (there $c \equiv \rho$).

2.5.1 Preliminaries

Whereas in the last section, we did not care about the specific form of the generator (as long as Dynkin's formula was valid), we now need to investigate the operator $\mathcal{L}^\beta$ more in detail:

$$\begin{aligned} \mathcal{L}^\beta u(x) = & \frac{1}{2} tr\left(\sigma(x,\beta)\sigma^T(x,\beta)D^2u(x)\right) + \langle \mu(x,\beta), \nabla u(x)\rangle \\ & + \int u(x + \ell(x,\beta,z)) - u(x) - \langle \nabla u(x), \ell(x,\beta,z)\rangle 1_{|z|<1}\, \nu(dz). \end{aligned} \tag{2.16}$$

We recall the definition of the function space $\mathcal{PB} = \mathcal{PB}(\mathbb{R}^d)$ from §2.2, such that the integro-differential operator $\mathcal{L}^\beta$ is well-defined for $\phi \in \mathcal{PB} \cap C^2(\mathbb{R}^d)$. Denoting for $0 < \delta < 1$, $y, p \in \mathbb{R}^d$, $X \in \mathbb{R}^{d\times d}$, $r \in \mathbb{R}$ and $(l_\beta)_{\beta\in B} \subset \mathbb{R}$:

$$\begin{aligned}
F(x, r, p, X, (l_\beta)) &= -\sup_{\beta\in B}\left\{\frac{1}{2}tr\left(\sigma(x,\beta)\sigma^T(x,\beta)X\right) + \langle\mu(x,\beta), p\rangle - c(x)r + f(x,\beta) + l_\beta\right\} \\
\mathcal{I}_\beta^{1,\delta}[x,\phi] &= \int_{|z|<\delta} \phi(x + \ell(x,\beta,z)) - \phi(x) - \langle\nabla\phi(x), \ell(x,\beta,z)\rangle\, \nu(dz) \\
\mathcal{I}_\beta^{2,\delta}[x,p,\phi] &= \int_{|z|\geq\delta} \phi(x + \ell(x,\beta,z)) - \phi(x) - \langle p, \ell(x,\beta,z)\rangle 1_{|z|<1}\, \nu(dz) \\
\mathcal{I}_\beta[x,\phi] &= \mathcal{I}_\beta^{1,\delta}[x,\phi] + \mathcal{I}_\beta^{2,\delta}[x,\nabla\phi(x),\phi],
\end{aligned}$$

we have to analyze the problem

$$\min(F(x, u(x), \nabla u(x), D^2u(x), \mathcal{I}_\beta[x, u(\cdot)]), u(x) - \mathcal{M}u(x)) = 0,$$

where the notation $u(\cdot)$ in the integral indicates that nonlocal terms are used on u, not only from x. As well, $\mathcal{I}_\beta[x, u(\cdot)]$ within F always stands for a family ($\beta \in B$) of integrals. Denote by F^β the function F without the sup, i.e. for a concrete β.

Remark 2.5.1. The following properties hold for our problem:

(P1) Ellipticity of F: $F(x, r, p, X^1, (l^1_\beta)) \leq F(x, r, p, X^2, (l^2_\beta))$ if $X^1 \geq X^2$, $l^1_\beta \geq l^2_\beta$ $\forall\beta \in B$

(P2) Translation invariance: $u - \mathcal{M}u = (u + l) - \mathcal{M}(u + l)$, $\mathcal{I}[y_0, \phi] = \mathcal{I}[y_0, \phi + l]$ for constants $l \in \mathbb{R}$

(P3) $(l_\beta)_\beta \mapsto F(x, r, p, X, (l_\beta))$ is continuous in the sense that

$$|F(x, r, p, X, (l^1_\beta)) - F(x, r, p, X, (l^2_\beta))| \leq \sup_\beta |l^1_\beta - l^2_\beta|.$$

The last statement — proved by easy sup manipulations — is just for the sake of completeness; we will not use it explicitly because uniform convergence needs continuous functions, which in general we do not have.

For $x \in S$ and $\delta > 0$, recall the definition of $U_\delta = U_\delta(x) = \{\ell(x, \beta, z) : |z| < \delta\}$. The integral containing the singular part, $\mathcal{I}_\beta^{1,\delta}[x, \phi]$, is only affected by a change of ϕ on $U_\delta(x)$.

Let henceforth Assumptions 2.2.1 and 2.2.3 be satisfied (the latter needed mainly for the equivalence of different viscosity solution definitions). Further assume

(U1*) c is continuous

Remark 2.5.2. It is sufficient for the comparison theorem if Assumption 2.2.3 holds only for small $\delta > 0$: "*For any x_0, there is a small neighbourhood and a $\bar{\delta} > 0$, where the assumption holds for $0 < \delta < \bar{\delta}\ldots$*" This is why we will carry out all proofs for ℓ depending on β in the following.

Immediately from (V3) and (U1*), it follows that $\sup_{\beta\in B} \sigma(x, \beta) < \infty$, and by sup manipulations that $(x, r, p, X) \mapsto F(x, r, p, X, l_\beta)$ is continuous; but even more can be deduced:

Proposition 2.5.1. *Let $(\beta_k) \subset B$ with $\beta_k \to \beta$, and $(x_k), (p_k) \subset \mathbb{R}^d$ with $x_k \to x \in S$, $p_k \to p$.*

(i) *If* $u \in \mathcal{PB} \cap USC(\mathbb{R}^d)$ *and* $v \in \mathcal{PB} \cap LSC(\mathbb{R}^d)$ *with* $u(x_k) \to u(x)$, $v(x_k) \to v(x)$, *then*

$$\limsup_{k\to\infty} \mathcal{I}^{2,\delta}_{\beta_k}[x_k, p_k, u(\cdot)] \leq \mathcal{I}^{2,\delta}_{\beta}[x, p, u(\cdot)], \qquad \liminf_{k\to\infty} \mathcal{I}^{2,\delta}_{\beta_k}[x_k, p_k, v(\cdot)] \geq \mathcal{I}^{2,\delta}_{\beta}[x, p, v(\cdot)].$$

Moreover, for $(\varphi_k), (\psi_k) \subset \mathcal{PB} \cap C^2(\mathbb{R}^d)$ *with* $\varphi_k \to u$ *and* $\psi_k \to v$ *monotonically,* $\varphi_k(x_k) \to u(x)$, $\psi_k(x_k) \to v(x)$,

$$\limsup_{k\to\infty} \mathcal{I}^{2,\delta}_{\beta_k}[x_k, p_k, \varphi_k(\cdot)] \leq \mathcal{I}^{2,\delta}_{\beta}[x, p, u(\cdot)], \qquad \liminf_{k\to\infty} \mathcal{I}^{2,\delta}_{\beta_k}[x_k, p_k, \psi_k(\cdot)] \geq \mathcal{I}^{2,\delta}_{\beta}[x, p, v(\cdot)].$$

(ii) *If* $\varphi \in C^2(\mathbb{R}^d)$, *then* $(x, \beta) \mapsto \mathcal{I}^{1,\delta}_{\beta}[x, \varphi(\cdot)]$ *is continuous. Moreover, for* $(\varphi_k) \subset C^2(\mathbb{R}^d)$ *with* $\varphi_k \to \varphi$ *monotonically,* $\varphi_k = \varphi$ *in a neighbourhood of* x,

$$\lim_{k\to\infty} \mathcal{I}^{1,\delta}_{\beta_k}[x_k, \varphi_k(\cdot)] = \mathcal{I}^{1,\delta}_{\beta}[x, \varphi(\cdot)].$$

(iii) *If* $u \in \mathcal{PB} \cap USC(\mathbb{R}^d)$ *and* $v \in \mathcal{PB} \cap LSC(\mathbb{R}^d)$, $(\varphi_k), (\psi_k) \subset \mathcal{PB} \cap C^2(\mathbb{R}^d)$ *with* $\varphi_k \to u$ *and* $\psi_k \to v$ *monotonically,* $\varphi_k = u \in C^2$ *and* $\psi_k = v \in C^2$ *in a neighbourhood of* x, *then*

$$\liminf_{k\to\infty} F^{\beta_k}(x, r, p, X, \mathcal{I}_{\beta_k}[x_k, \varphi_k(\cdot)]) \geq F^{\beta}(x, r, p, X, \mathcal{I}_{\beta}[x, u(\cdot)]) \tag{2.35}$$

$$\limsup_{k\to\infty} F^{\beta_k}(x, r, p, X, \mathcal{I}_{\beta_k}[x_k, \psi_k(\cdot)]) \leq F^{\beta}(x, r, p, X, \mathcal{I}_{\beta}[x, v(\cdot)]). \tag{2.36}$$

(iv) *If* $u \in \mathcal{PB} \cap USC(\mathbb{R}^d)$, $\varphi \in C^2(\mathbb{R}^d)$, *then* $\beta \mapsto -F^{\beta}(x, r, p, X, \mathcal{I}^{1,\delta}_{\beta}[x, \varphi(\cdot)] + \mathcal{I}^{2,\delta}_{\beta}[x, p, u(\cdot)]$ *is in* $USC(B)$. *In particular, the supremum in* $\beta \in B$ *is assumed.*

Proof: (i): We prove only the first statement for $u \in USC$ (the lsc proof being analogous). By a general version of the dominated convergence theorem and the definition of $\mathcal{PB}$,

$$\begin{aligned} &\limsup_{k\to\infty} \int_{|z|\geq\delta} u(x_k + \ell(x_k, \beta_k, z)) - u(x_k) - \langle p_k, \ell(x_k, \beta_k, z)\rangle 1_{|z|<1}\, \nu(dz) \\ \leq\ & \int_{|z|\geq\delta} \limsup_{k\to\infty}\, u(x_k + \ell(x_k, \beta_k, z)) - u(x_k) - \langle p_k, \ell(x_k, \beta_k, z)\rangle 1_{|z|\leq 1}\, \nu(dz) \\ \leq\ & \int_{|z|\geq\delta} u(x + \ell(x, \beta, z)) - u(x) - \langle p, \ell(x, \beta, z)\rangle 1_{|z|<1}\, \nu(dz), \end{aligned} \tag{2.37}$$

where we have used the continuity of ℓ (V3). The fact for φ_k follows because by monotonicity of (φ_k), and a general version of Dini's theorem (cf. DiBenedetto [38], Th. 7.3), $\limsup_{k\to\infty} \varphi_k(x_k + \ell(x_k, \beta_k, z)) \leq u^*(x + \ell(x, \beta, z))$.
(ii): Outside of the singularity of ν, the result follows from (i). In the neighbourhood of x, the Taylor expansion (2.14) gives the upper bound for the application of the DCT (where the local boundedness of $U_\delta(x)$ holds by (V4)).
(iii) follows immediately from (i), (ii). Finally, (iv) holds by the continuity conditions (V3), (U1*) and (i), (ii). □

2.5.2 Viscosity solutions: Different definitions

Let us now restate in the new notation the definition of viscosity solution in the elliptic case (equivalent thanks to the translation invariance property):

Definition 2.5.2 (Viscosity solution 1)**.** *A function $u \in \mathcal{PB}$ is a (viscosity) subsolution of (2.34) if for all $x_0 \in \mathbb{R}^d$ and $\varphi \in \mathcal{PB} \cap C^2(\mathbb{R}^d)$ such that $u^* - \varphi$ has a global maximum in x_0,*

$$\begin{aligned} \min\left(F(x_0, u^*, \nabla\varphi, D^2\varphi, \mathcal{I}_\beta[x_0, \varphi(\cdot)]), u^* - \mathcal{M}u^*\right) &\leq 0 \qquad \text{in } x_0 \in S \\ \min\left(u^* - g, u^* - \mathcal{M}u^*\right) &\leq 0 \qquad \text{in } x_0 \in (\mathbb{R}^d \setminus S). \end{aligned}$$

A function $u \in \mathcal{PB}$ is a (viscosity) supersolution of (2.34) if for all $x_0 \in \mathbb{R}^d$ and $\varphi \in \mathcal{PB} \cap C^2(\mathbb{R}^d)$ such that $u_* - \varphi$ has a global minimum in x_0,

$$\begin{aligned} \min\left(F(x_0, u_*, \nabla\varphi, D^2\varphi, \mathcal{I}_\beta[x_0, \varphi(\cdot)]), u_* - \mathcal{M}u_*\right) &\geq 0 \qquad \text{in } x_0 \in S \\ \min\left(u_* - g, u_* - \mathcal{M}u_*\right) &\geq 0 \qquad \text{in } x_0 \in (\mathbb{R}^d \setminus S). \end{aligned}$$

A function u is a viscosity solution if it is sub and supersolution.

Equivalently, the definition can be formulated with "strict global maximum" and "strict global minimum", respectively. Indeed, one direction is obvious. For the inverse direction, consider $\varphi_\alpha := \varphi + \alpha\chi$ for a function $\chi \in \mathcal{PB} \cap C^2(\mathbb{R}^d)$ with $\chi > 0$ on $\mathbb{R}^d \setminus \{x_0\}$, and $\chi(x_0) = 0$, $\nabla\chi(x_0) = 0$, $D^2\chi(x_0) = 0$. If $u^* - \varphi$ has a global maximum in x_0, then $u^* - \varphi_\alpha$ has a strict global maximum in x_0 for any $\alpha > 0$. By results very similar to Proposition 2.5.1, the mapping $(\alpha, \beta) \mapsto F^\beta(x_0, r, p, X, I_\beta[x_0, \varphi_\alpha(\cdot)])$ is continuous, and thus also $\alpha \mapsto F(x_0, r, p, X, I_\beta[x_0, \varphi_\alpha(\cdot)])$ is continuous as sup over the compact set B; compare also the proof of Corollary 2.5.9. Thus the equivalence holds by taking the limit $\alpha \to 0$.

As in [11], we give two further equivalent definitions, of which the latter one is needed later on in the uniqueness proof:

Definition 2.5.3 (Viscosity solution 2)**.** *A function $u \in \mathcal{PB}$ is a (viscosity) subsolution of (2.34) if for all $x_0 \in \mathbb{R}^d$ and $\varphi \in C^2(\mathbb{R}^d)$ such that $u^* - \varphi$ has a maximum in x_0 on $U_\delta(x_0)$,*

$$\begin{aligned} \min\Big(F(x_0, u^*, \nabla\varphi, D^2\varphi, \mathcal{I}^{1,\delta}_\beta[x_0, \varphi(\cdot)] + \mathcal{I}^{2,\delta}_\beta[x_0, \nabla\varphi, u^*(\cdot)]), u^* - \mathcal{M}u^*\Big) &\leq 0 \qquad \text{in } x_0 \in S \\ \min\left(u^* - g, u^* - \mathcal{M}u^*\right) &\leq 0 \qquad \text{in } \mathbb{R}^d \setminus S. \end{aligned}$$

A function $u \in \mathcal{PB}$ is a (viscosity) supersolution of (2.34) if for all $x_0 \in \mathbb{R}^d$ and $\varphi \in C^2(\mathbb{R}^d)$ such that $u_* - \varphi$ has a minimum in x_0 on $U_\delta(x_0)$,

$$\begin{aligned} \min\Big(F(x_0, u_*, \nabla\varphi, D^2\varphi, \mathcal{I}^{1,\delta}_\beta[x_0, \varphi(\cdot)] + \mathcal{I}^{2,\delta}_\beta[x_0, \nabla\varphi, u_*(\cdot)]), u_* - \mathcal{M}u_*\Big) &\geq 0 \qquad \text{in } x_0 \in S \\ \min\left(u_* - g, u_* - \mathcal{M}u_*\right) &\geq 0 \qquad \text{in } \mathbb{R}^d \setminus S. \end{aligned}$$

A function u is a viscosity solution if it is sub and supersolution.

Note that the definition is of course still valid if $U_\delta(x_0) \cap (\mathbb{R}^d \setminus S) \neq \emptyset$. That the two definitions are equivalent is essentially not a new result; for proofs in simpler settings, see Alvarez and Tourin [5], Jakobsen and Karlsen [68].

Proposition 2.5.4. *Definitions 1 and 2 are equivalent.*

Proof: We treat first the subsolution part. Because $u^* - \mathcal{M}u^* \leq 0$ holds independently of the viscosity formulation, we need only consider the PIDE part for $x_0 \in S$. (Note that for $\nu(\mathbb{R}^k) < \infty$, we can set $\delta = 0$, and use in the following an arbitrary local neighbourhood of x_0 instead of $U_\delta(x_0)$.)

"$\Leftarrow$" Assume u is a viscosity subsolution according to Definition 2. Let $x_0 \in S$ and $\varphi \in \mathcal{PB} \cap C^2(\mathbb{R}^d)$ such that $u^* - \varphi$ has a global maximum in x_0. Then x_0 is also a maximum point on $U_\delta(x_0)$. So

$$F(x_0, u^*, \nabla\varphi, D^2\varphi, \mathcal{I}^{1,\delta}_\beta[x_0, \varphi(\cdot)] + \mathcal{I}^{2,\delta}_\beta[x_0, \nabla\varphi, u^*(\cdot)]) \leq 0.$$

By the property $\mathcal{I}^{2,\delta}_\beta[x_0, \nabla\varphi, u^*(\cdot)] \leq \mathcal{I}^{2,\delta}_\beta[x_0, \nabla\varphi, \varphi(\cdot)]$, and ellipticity of F, we are done.

"$\Rightarrow$" Assume u is a viscosity subsolution according to Definition 1. Let $x_0 \in S$ and $\varphi \in C^2(\mathbb{R}^d)$ such that $u^* - \varphi$ has a maximum in x_0 on $U_\delta(x_0)$. Wlog assume that $u^*(x_0) = \varphi(x_0)$, $u^* \leq \varphi$ on $U_\delta(x_0)$.

Now consider the function

$$\psi(x) := 1_{\mathbb{R}^d \setminus U_\delta}(x) u^*(x) + 1_{U_\delta}(x)\varphi(x),$$

which is in $\mathcal{PB}$. It is immediate that ψ is upper semicontinuous if U_δ is closed, and in this case, we can construct a monotonically decreasing sequence (e.g., by approximating with piecewise constant functions, smoothed by the standard mollifier) $(\varphi_k) \subset \mathcal{PB} \cap C^2(\mathbb{R}^d)$ such that $\varphi_k \downarrow \psi$ pointwise.

If U_δ is not closed, then ψ need not be upper semicontinuous (if $u^*(y) < \varphi(y)$ for a $y \in \partial U_\delta \cap (U_\delta)^c$). In this case however, there is a small open neighourhood V of y where $u^* < \varphi$ (because $\varphi \in LSC$, $u^* \in USC$). So we can approximate φ from below in $V \cap int(U_\delta)$ with $\varphi_k \geq u^*$, and combine this in the construction as above.

Let further $\varphi_k = \varphi$ in $B(x_0, \rho) \subset\subset U_\delta(x_0)$ for some $\rho > 0$ (possible by (U2)). Then $u^*(x_0) = \varphi_k(x_0)$, $u^* \leq \varphi_k$, and all local properties for φ_k in x_0 are inherited from φ. Applying Definition 1 yields

$$F(x_0, u^*, \nabla\varphi, D^2\varphi, \mathcal{I}^{1,\delta}_\beta[x_0, \varphi_k(\cdot)] + \mathcal{I}^{2,\delta}_\beta[x_0, \nabla\varphi, \varphi_k(\cdot)]) \leq 0.$$

For each k, by Proposition 2.5.1 (iv), the supremum in F is attained in a β_k. Choose a subsequence such that $\beta_k \to \beta$. Using the limit in (2.35), we obtain our result.

Supersolution part, "$\Rightarrow$": For analogously defined φ, here we have to show that

$$F(x_0, u_*, \nabla\varphi, D^2\varphi, \mathcal{I}^{1,\delta}_\beta[x_0, \varphi(\cdot)] + \mathcal{I}^{2,\delta}_\beta[x_0, \nabla\varphi, u_*(\cdot)]) \geq 0,$$

for which it is sufficient to prove $F^\beta(\dots) \geq 0$ for all $\beta \in B$. We construct the sequence (φ_k) with $\varphi_k \to \psi$,

$$\psi(x) := 1_{\mathbb{R}^d \setminus U_\delta}(x) u_*(x) + 1_{U_\delta}(x)\varphi(x),$$

as in the subsolution case (where the problematic part is $u_*(y) > \varphi(y)$ in a $y \in \partial U_\delta \cap (U_\delta)^c$). The result is obtained analogously (with β fixed) to the subsolution case using (2.36). □

Remark 2.5.3. The approximation from above of a usc function by C^2 functions used in the second part of the proof of Proposition 2.5.4 shows also that, in Definitions 1 and 2, we may restrict ourselves to functions $\varphi \in C^0(\mathbb{R}^d)$ that are only C^2 in a small neighbourhood of x_0.

We recall the semijets needed for a third equivalent definition. They are motivated by a classical property of differentiable functions. Let $u : \mathbb{R}^d \to \mathbb{R}$.

$$J^+u(x) = \{(p, X) \in \mathbb{R}^d \times \mathbb{S}^d : u(x+z) \leq u(x) + \langle p, z\rangle + \frac{1}{2}\langle Xz, z\rangle + o(|z|^2) \text{ as } z \to 0\}$$
$$J^-u(x) = \{(p, X) \in \mathbb{R}^d \times \mathbb{S}^d : u(x+z) \geq u(x) + \langle p, z\rangle + \frac{1}{2}\langle Xz, z\rangle + o(|z|^2) \text{ as } z \to 0\}$$

If u is twice differentiable at x, then $J^+u(x) \cap J^-u(x) = \{(\nabla u(x), D^2u(x))\}$. The limiting semijets are defined by, e.g.,

$$\overline{J}^+u(x) = \{(p, X) \in \mathbb{R}^d \times \mathbb{S}^d : \text{ there exist } (x_k, p_k, X_k) \to (x, p, X), \\ (p_k, X_k) \in J^+u(x_k) \text{ such that } u(x_k) \to u(x)\}.$$

Definition 2.5.5 (Viscosity solution 3)**.** *A function $u \in \mathcal{PB}$ is a (viscosity) subsolution of (2.34) if for all $x_0 \in \mathbb{R}^d$ and $\varphi \in C^2(\mathbb{R}^d)$ such that $u^* - \varphi$ has a maximum in x_0 on $U_\delta(x_0)$ and for $(p, X) \in J^+u^*(x_0)$ with $p = D\varphi(x_0)$ and $X \le D^2\varphi(x_0)$,*

$$\begin{aligned} \min\Big(F(x_0, u^*, p, X, \mathcal{I}_\beta^{1,\delta}[x_0, \varphi(\cdot)] + \mathcal{I}_\beta^{2,\delta}[x_0, p, u^*(\cdot)]), u^* - \mathcal{M}u^*\Big) &\le 0 \qquad \textit{in } x_0 \in S \\ \min(u^* - g, u^* - \mathcal{M}u^*) &\le 0 \qquad \textit{in } x_0 \in (\mathbb{R}^d \setminus S). \end{aligned}$$

A function $u \in \mathcal{PB}$ is a (viscosity) supersolution of (2.34) if for all $x_0 \in \mathbb{R}^d$ and $\varphi \in C^2(\mathbb{R}^d)$ such that $u_* - \varphi$ has a minimum in x_0 on $U_\delta(x_0)$ and for $(q, Y) \in J^-u_*(x_0)$ with $q = D\varphi(x_0)$ and $Y \ge D^2\varphi(x_0)$,

$$\begin{aligned} \min\Big(F(x_0, u_*, q, Y, \mathcal{I}_\beta^{1,\delta}[x_0, \varphi(\cdot)] + \mathcal{I}_\beta^{2,\delta}[x_0, q, u_*(\cdot)]), u_* - \mathcal{M}u_*\Big) &\ge 0 \qquad \textit{in } x_0 \in S \\ \min(u_* - g, u_* - \mathcal{M}u_*) &\ge 0 \qquad \textit{in } x_0 \in (\mathbb{R}^d \setminus S). \end{aligned}$$

A function u is a viscosity solution if it is sub and supersolution.

The conditions $p = D\varphi(x_0)$ and $X \le D^2\varphi(x_0)$ etc. and the maximum condition seem to be superfluous at first view. However, they are needed to ensure consistency of φ with the "local" derivatives (p, X).

Proposition 2.5.6. *Definitions 2 and 3 are equivalent.*

Proof: One direction is obvious, using the properties of J^+, J^-. The other direction (see [11]) uses as vital ingredient that we are considering the *local* maximum. □

2.5.3 A maximum principle

Following [11] we give here a nonlocal theorem which should replace the "maximum principle for semicontinuous functions", as stated in Theorem 1.2.1. Prior to this, we have to collect some properties of the intervention operator $\mathcal{M}$ (compare also Lemma 2.4.3):

Lemma 2.5.7. *(i) $\mathcal{M}$ is convex, i.e. for $\lambda \in [0, 1]$, $\mathcal{M}(\lambda a + (1-\lambda)b) \le \lambda\mathcal{M}a + (1-\lambda)\mathcal{M}b$.*

(ii) For $\lambda > 0$, $\mathcal{M}(-\lambda a + (1+\lambda)b) \ge -\lambda\mathcal{M}a + (1+\lambda)\mathcal{M}b$ (assuming the latter is not $\infty - \infty$).

Proof: Follows easily from $\sup_x(a(x) + b(x)) \le \sup_x a(x) + \sup_x b(x)$ and $\sup_x(a(x) + b(x)) \ge \sup_x a(x) + \inf_x b(x)$, respectively. □

We need the following nonlocal Jensen-Ishii lemma that can be applied in the PIDE case (compare discussion below):

Lemma 2.5.8 (Lemma 1 in [11]). *Let $u \in USC(\mathbb{R}^d)$ and $v \in LSC(\mathbb{R}^d)$, $\varphi \in C^2(\mathbb{R}^{2d})$. If $(x_0, y_0) \in \mathbb{R}^{2d}$ is a zero global maximum point of $u(x) - v(y) - \varphi(x, y)$ and if $p = D_x\varphi(x_0, y_0)$, $q = D_y\varphi(x_0, y_0)$, then for any $K > 0$, there exists $\bar{\alpha}(K) > 0$ such that, for any $0 < \alpha < \bar{\alpha}(K)$, we have: There exist sequences $x_k \to x_0$, $y_k \to y_0$, $p_k \to p$, $q_k \to q$, matrices X_k, Y_k and a sequence of functions (φ_k), converging to the function $\varphi_\alpha(x, y) := R^\alpha[\varphi]((x, y), (p, q))$ uniformly in $\mathbb{R}^{2d}$ and in $C^2(B((x_0, y_0), K))$, such that*

$$u(x_k) \to u(x_0), v(y_k) \to v(y_0) \tag{2.38}$$

$$(x_k, y_k) \text{ is a global maximum point of } u - v - \varphi_k \tag{2.39}$$

$$(p_k, X_k) \in J^+u(x_k) \tag{2.40}$$

$$(q_k, Y_k) \in J^-v(y_k) \tag{2.41}$$

$$-\frac{1}{\alpha} I \leq \begin{bmatrix} X_k & 0 \\ 0 & -Y_k \end{bmatrix} \leq D^2\varphi_k(x_k, y_k). \tag{2.42}$$

Here $p_k = \nabla_x\varphi_k(x_k, y_k)$, $q_k = \nabla_y\varphi_k(x_k, y_k)$, and $\varphi_\alpha(x_0, y_0) = \varphi(x_0, y_0)$, $\nabla\varphi_\alpha(x_0, y_0) = \nabla\varphi(x_0, y_0)$.

Remark 2.5.4. The expression $\varphi_\alpha(x, y) = R^\alpha[\varphi]((x, y), (p, q))$ is the "modified sup-convolution" as used by Barles and Imbert [11]. For all compacts C, φ_α converges uniformly to φ in $C^2(C)$ as $\alpha \to 0$. This was already used in [11], and can be seen by classical arguments using the implicit function theorem.

We would obtain a variant of the local Jensen-Ishii lemma (also called maximum principle), if we weren't interested in the sequence (φ_k) converging in C^2 — in this case the statement could be expressed in terms of the *limiting* semijets (or "closures") $\overline{J}^+$, $\overline{J}^-$ (e.g., $(p, X) \in \overline{J}^+u(x_0)$). However, the local Jensen-Ishii lemma is only useful (in the PDE case), because it can be directly used to deduce, e.g., $F(x_0, u^*(x_0), p, X) \leq 0$ by continuity of F. Compare also the more detailed explanation in Jakobsen and Karlsen [68].

This immediate consequence in the PDE case is a bit more tedious to show in our PIDE case (because the Lévy measure ν is possibly singular at 0), and needs the C^2 convergence of the (φ_k). The corollary for our impulse control purposes takes the following form:

Corollary 2.5.9. *Assume (V1), (V2). Let u be a viscosity subsolution and v a viscosity supersolution of (2.34), and $\varphi \in C^2(\mathbb{R}^{2d})$. If $(x_0, y_0) \in \mathbb{R}^{2d}$ is a global maximum point of $u^*(x) - v_*(y) - \varphi(x, y)$, then, for any $\delta > 0$, there exists $\bar{\alpha}$ such that for $0 < \alpha < \bar{\alpha}$, there are $(p, X) \in \overline{J}^+u^*(x_0)$ and $(q, Y) \in \overline{J}^-v_*(y_0)$ with*

$$\begin{aligned} \min\Big(F(x_0, u^*(x_0), p, X, \mathcal{I}^{1,\delta}_\beta[x_0, \varphi_\alpha(\cdot, y_0)] + \mathcal{I}^{2,\delta}_\beta[x_0, p, u^*(\cdot)]), u^* - \mathcal{M}u^*\Big) &\leq 0 \\ \min\Big(F(y_0, v_*(y_0), q, Y, \mathcal{I}^{1,\delta}_\beta[y_0, -\varphi_\alpha(x_0, \cdot)] + \mathcal{I}^{2,\delta}_\beta[y_0, q, v_*(\cdot)]), v_* - \mathcal{M}v_*\Big) &\geq 0 \end{aligned}$$

if $x_0 \in S$ or $y_0 \in S$, respectively. Here $p = \nabla_x\varphi(x_0, y_0) = \nabla_x\varphi_\alpha(x_0, y_0)$, $q = -\nabla_y\varphi(x_0, y_0) = -\nabla_y\varphi_\alpha(x_0, y_0)$, and furthermore,

$$-\frac{1}{\alpha} I \leq \begin{bmatrix} X & 0 \\ 0 & -Y \end{bmatrix} \leq D^2\varphi_\alpha(x_0, y_0) = D^2\varphi(x_0, y_0) + o_\alpha(1). \tag{2.43}$$

If none of x_0, y_0 is in S, then we do not need this corollary, because the boundary conditions are not dependent on derivatives of functions φ.

Remark 2.5.5. Note that the fact $(p, X) \in \overline{J}^+ u(x_0)$ (and the corresponding for the supersolution) is not needed in the statement of the corollary, because the subsolution (supersolution) inequality directly holds by the approximation procedure in the proof. An abstract way of formulating Lemma 2.5.8 and Corollary 2.5.9 in the style of the local Jensen-Ishii Lemma (only subsolution without impulses) would be to define a new "limiting superjet" containing the (p, X, φ_α) obtained as limit of the terms in Lemma 2.5.8. Then Corollary 2.5.9 could be stated as "*For* (p, X, φ_α) *in the limiting superjet,* $F(x_0, u^*(x_0), p, X, \varphi_\alpha(\cdot), u^*(\cdot)) \leq 0$" and would follow directly from Lemma 2.5.8, provided some "continuity" of $F : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d \times C^2 \times \mathcal{PB} \to \mathbb{R}$ hold.

Proof of Corollary 2.5.9: Because of translation invariance, we can assume wlog that $u^*(x_0) - v_*(y_0) - \varphi(x_0, y_0) = 0$. Choose sequences according to Lemma 2.5.8 (applied for u^* and v_*), and $K := \max(dist(x_0, U_\delta(x_0)), dist(y_0, U_\delta(y_0))) + 1$. Fix $\alpha \in (0, \bar{\alpha}(K))$.

Subsolution case: If $x_0 \in S$, then $x_k \in S$ for k large, and by Definition 3 and (2.39)-(2.42),

$$\min\Big(F(x_k, u^*(x_k), p_k, X_k, \mathcal{I}^{1,\delta}_\beta[x_k, \varphi_k(\cdot, y_k)] + \mathcal{I}^{2,\delta}_\beta[x_k, p_k, u^*(\cdot)]), u^*(x_k) - \mathcal{M}u^*(x_k)\Big) \leq 0. \tag{2.44}$$

First let us prove convergence of the PIDE part F. First, $x_k \to x_0$, $u^*(x_k) \to u^*(x_0)$, $p_k \to p$ by Lemma 2.5.8. (X_k) is contained in a compact set in $\mathbb{R}^{d\times d}$ by (2.42), so it admits a convergent subsequence to an X satisfying (2.43). For each k, by Proposition 2.5.1 (iv), the supremum in F is attained in a β_k. Choose another (sub-)subsequence converging to $\beta \in B$.
We now only need a reinforced version of (2.35) in Proposition 2.5.1 for $\mathcal{I}^{1,\delta}_\beta$. By Lebesgue's dominated convergence theorem, (V3) and uniform convergence,

$$\begin{aligned} &\lim_{k\to\infty} \int_{|z|<\delta} \varphi_k(x_k + \ell(x_k, \beta_k, z)) - \varphi_k(x_k) - \langle \nabla\varphi_k(x_k), \ell(x_k, \beta_k, z)\rangle\, \nu(dz) \\ = &\int_{|z|<\delta} \varphi_\alpha(x_0 + \ell(x_0, \beta, z)) - \varphi_\alpha(x_0) - \langle \nabla\varphi_\alpha(x_0), \ell(x_k, \beta, z)\rangle\, \nu(dz), \end{aligned}$$

where the ν-integrable upper estimate can be derived by Taylor expansion and the estimates for k large

$$\sup_{|z-x|<\kappa_1} |D^2\varphi_k(z)| \leq \sup_{|z-x|<\kappa_1} |D^2\varphi_\alpha(z)| + \kappa_2$$

for some $\kappa_1, \kappa_2 > 0$ (recall that $\int_C |z|^2\nu(dz) < \infty$ for all compacts C, and that $U_\delta(y_0) \downarrow 0$ for singular ν). For $\mathcal{I}^{2,\delta}_\beta$, we use Proposition 2.5.1 (i). For the impulse part, we know by Lemma 2.4.3 (ii) that $\mathcal{M}u^*$ is usc, so

$$\liminf_{k\to\infty} u^*(x_k) - \mathcal{M}u^*(x_k) = u^*(x_0) - \limsup_{k\to\infty} \mathcal{M}u^*(x_k) \geq u^*(x_0) - \mathcal{M}u^*(x_0).$$

Now we have to combine the estimates derived so far. By iteratively taking subsequences and using (2.35), we have the desired result for $k \to \infty$ in (2.44).

Supersolution case: If $y_0 \in S$, then $y_k \in S$ for k large and by Definition 3 and (2.39)-(2.42),

$$\min\Big(F(y_k, v_*(y_k), q_k, Y_k, \mathcal{I}^{1,\delta}_\beta[y_k, -\varphi_k(x_k, \cdot)] + \mathcal{I}^{2,\delta}_\beta[y_k, p_k, v_*(\cdot)]), v_*(y_k) - \mathcal{M}v_*(y_k)\Big) \geq 0,$$

which means that two separate inequalities for the PIDE part and for the impulse part hold. The convergence of the PIDE part is proved in a completely analogous way, except that now

(2.36) in Proposition 2.5.1 is used in a reinforced version (again only needed for $\beta \in B$ fixed). For the impulse part, we know that $\mathcal{M}v_*$ is lsc by Lemma 2.4.3 (i), so

$$\limsup_{k\to\infty} v_*(y_k) - \mathcal{M}v_*(y_k) = v_*(y_0) - \liminf_{k\to\infty} \mathcal{M}v_*(y_k) \le v_*(x_0) - \mathcal{M}v_*(y_0).$$

□

Remark 2.5.6. By inspecting the proof of Corollary 2.5.9, we see that the statement also holds if u and v are subsolution and supersolution, respectively, of different HJBQVIs, provided of course that the conditions are satisfied. This will be used in the proof of the comparison Theorem 2.5.11.

2.5.4 A comparison result

Now we are prepared to give a comparison result (inspired by Ishii [65]):

Lemma 2.5.10. *Assume (V1), (V2). Let u be a subsolution and v a supersolution of (2.34), further assume that there is a $w \in \mathcal{PB} \cap C^2(\mathbb{R}^d)$ and a positive function $\kappa : \mathbb{R}^d \to \mathbb{R}$ such that*

$$\begin{aligned} \min(-\sup_{\beta\in B}\{-cw + \mathcal{L}^\beta w + f^\beta\}, w - \mathcal{M}w) &\ge \kappa \qquad \text{in } S \\ \min(w - g, w - \mathcal{M}w) &\ge \kappa \qquad \text{in } \mathbb{R}^d \setminus S. \end{aligned}$$

Then for $m \in \mathbb{N}$, $v_m := (1 - \frac{1}{m})v + \frac{1}{m}w$ is a supersolution of

$$\begin{aligned} \min(-\sup_{\beta\in B}\{-cu + \mathcal{L}^\beta u + f^\beta\}, u - \mathcal{M}u) - \kappa/m &= 0 \qquad \text{in } S \\ \min(u - g, u - \mathcal{M}u) - \kappa/m &= 0 \qquad \text{in } \mathbb{R}^d \setminus S, \end{aligned} \tag{2.45}$$

and $u_m := (1 + \frac{1}{m})u - \frac{1}{m}w$ is a subsolution of (2.34), and of (2.45) with $-\kappa$ replaced by $+\kappa$.

Proof: We use the first viscosity solution definition. Denote for the proof $\tilde{\mathcal{L}}^\beta u = -cu + \mathcal{L}^\beta u$. First consider the supersolution case. For ease of notation, we write v instead of v_*, and so on. Let $\varphi_m \in \mathcal{PB} \cap C^2(\mathbb{R}^d)$, $x_0 \in S$ such that $\varphi_m(x_0) = v_m(x_0)$, $\varphi_m \le v_m$. Choose $\varphi = (\varphi_m - \frac{1}{m}w)(\frac{m}{m-1})$, then $\varphi(x_0) = v(x_0)$ and $\varphi \le v$. We know that $-\sup_{\beta\in B}\{\tilde{\mathcal{L}}^\beta\varphi + f^\beta\} \ge 0$, and it is sufficient to show that $\tilde{\mathcal{L}}^\beta\varphi_m + f^\beta \le -\kappa/m$ for all $\beta \in B$, in a point $x_0 \in S$. Using the linearity of $\tilde{\mathcal{L}}$, we obtain

$$0 \ge \tilde{\mathcal{L}}^\beta(\varphi_m - \frac{1}{m}w) + \frac{m-1}{m}f^\beta = \tilde{\mathcal{L}}^\beta\varphi_m + f^\beta - \frac{1}{m}(\tilde{\mathcal{L}}^\beta w + f^\beta) \ge \tilde{\mathcal{L}}^\beta\varphi_m + f^\beta + \frac{\kappa}{m}.$$

Because of the convexity of $\mathcal{M}$ (Lemma 2.5.7 (i)), in any point $x_0 \in \mathbb{R}^d$,

$$\begin{aligned} v_m - \mathcal{M}v_m &\ge v_m - (1 - \frac{1}{m})\mathcal{M}v - \frac{1}{m}\mathcal{M}w \ge v_m - (1 - \frac{1}{m})v - \frac{1}{m}\mathcal{M}w \\ &= \frac{1}{m}(w - \mathcal{M}w) > \frac{\kappa}{m}. \end{aligned}$$

It is easy to check that $v_m - g \ge \frac{\kappa}{m}$.

For the subsolution u, the proof proceeds by a case distinction. The reasoning in the impulse part is the same, except that now the anticonvexity of $\mathcal{M}$ (Lemma 2.5.7 (ii)) is used. The PIDE part can be seen (for $\varphi_m(x_0) = u_m(x_0)$, $\varphi_m \geq v_m$, $\varphi = (\varphi_m + \frac{1}{m}w)(\frac{m}{m+1})$) by

$$0 \leq \frac{m+1}{m} \sup_\beta [\tilde{\mathcal{L}}^\beta \varphi + f^\beta] = \sup_\beta [\tilde{\mathcal{L}}^\beta \varphi_m + f^\beta + \frac{1}{m}(\tilde{\mathcal{L}}^\beta w + f^\beta)]$$
$$\leq \sup_\beta [\tilde{\mathcal{L}}^\beta \varphi_m + f^\beta] + \sup_\beta [\frac{1}{m}(\tilde{\mathcal{L}}^\beta w + f^\beta)] \leq \sup_\beta [\tilde{\mathcal{L}}^\beta \varphi_m + f^\beta] - \frac{\kappa}{m},$$

so we can conclude $-\sup_\beta [\tilde{\mathcal{L}}^\beta \varphi_m + f^\beta] \leq -\frac{\kappa}{m}$. □

We are going to use the perturbations of sub and supersolutions to make sure that the maximum of $u_m - v_m$ is attained. So we want to find a $w \geq 0$ growing faster than $|u|$ and $|v|$ as $|x| \to \infty$. How to find such a w is discussed in §2.2; the requirements lead to the function F being proper in the sense of Crandall et al. [33].

If $\sigma(\cdot, \beta)$, $\mu(\cdot, \beta)$, $f(\cdot, \beta)$, c are Lipschitz continuous, then by classical results (see, e.g., Lemma V.7.1 in Fleming and Soner [46]), our function F has the property:

> For any $R > 0$, there exists a modulus of continuity ω_R, such that, for any $|x|, |y|, |v| \leq R$, $l \in \mathbb{R}$ and for any $X, Y \in S^d$ satisfying
>
> $$\begin{bmatrix} X & 0 \\ 0 & -Y \end{bmatrix} \leq \frac{1}{\varepsilon} \begin{bmatrix} I & -I \\ -I & I \end{bmatrix} + o_\alpha(1)$$
>
> for some $\varepsilon > 0$ ($o_\alpha(1)$ does not depend on ε), then
>
> $$F(y, v, \varepsilon^{-1}(x-y), Y, l) - F(x, v, \varepsilon^{-1}(x-y), X, l) \leq \omega_R(|x-y| + \varepsilon^{-1}|x-y|^2) + o_\alpha(1),$$

where $o_\alpha(1)$ again does not depend on ε, and the first term is independent of α. In the proof of Theorem 2.5.11, x_ε, y_ε converge to the same limit x_0, so the requirement can be relaxed to locally Lipschitz (as required in Assumption 2.2.4) and the property holds only for x, y in a suitable neighbourhood of x_0.

Recall that $\mathcal{PB}_p$ is the space of functions in $\mathcal{PB}$ at most polynomially growing with exponent p. Assuming essentially that there is a strict supersolution of (2.34), the following theorem holds:

Theorem 2.5.11. *Let Assumptions 2.2.1, 2.2.3 and 2.2.4 be satisfied and c be locally Lipschitz continuous. Assume further that there is a $w \geq 0$ as in Lemma 2.5.10 (for a constant $\kappa > 0$) with $w(x)/|x|^p \to \infty$ for $|x| \to \infty$. If $u \in \mathcal{PB}_p(\mathbb{R}^d)$ is a subsolution and $v \in \mathcal{PB}_p(\mathbb{R}^d)$ a supersolution of (2.34), then $u^* \leq v_*$.*

Corollary 2.5.12 (HJBQVI viscosity solution: uniqueness)**.** *Under the same assumptions, there is at most one viscosity solution in $\mathcal{PB}_p(\mathbb{R}^d)$ of (2.34), and it is continuous.*

Proof: Let u_1, u_2 be two viscosity solutions of (2.34). Then by Th. 2.5.11 and definition of the upper and lower semicontinuous envelopes,

$$u_1^* \leq (u_2)_* \leq u_2^* \leq (u_1)_* \leq u_1^*,$$

and thus $u_1 = u_2$. □

The now following proof of Theorem 2.5.11 uses the strict sub/supersolution technique (adapted from Ishii [65]). We first prove that a maximum can not be attained outside S (because then this would have been because of an impulse back to S). Then we use the classical doubling of variables technique, apply the non-local maximum principle, and by a case distinction reduce the problem to a PIDE without impulse part (then adapting techniques of Barles and Imbert [11]).

Proof of Theorem 2.5.11: Write u instead of u^* and v instead of v_* to make the notation more convenient. It is sufficient to prove that $u_m - v_m \leq 0$ for all m large (where u_m, v_m are as defined in Lemma 2.5.10). Let $m \in \mathbb{N}$ be fixed for the moment. To prove by contradiction, let us assume that $M := \sup_{x\in\mathbb{R}^d} u_m(x) - v_m(x) > 0$.

Step 1. We want to show that the supremum can not be approximated from within $\mathbb{R}^d \setminus S$. Assume that for each $\varepsilon_1 > 0$, we can find an $\hat{x} = \hat{x}_{\varepsilon_1} \in \mathbb{R}^d \setminus S$ such that $u_m(\hat{x}) - v_m(\hat{x}) + \varepsilon_1 > M$ (and wlog $u_m(\hat{x}) - v_m(\hat{x}) > 0$). By the sub and supersolution definition, we have

$$\min(u_m(\hat{x}) - g(\hat{x}), u_m(\hat{x}) - \mathcal{M}u_m(\hat{x})) \leq 0$$
$$\min(v_m(\hat{x}) - g(\hat{x}), v_m(\hat{x}) - \mathcal{M}v_m(\hat{x})) \geq \kappa/m.$$

If $u_m(\hat{x}) - g(\hat{x}) \leq 0$, then $\kappa/m + u_m(\hat{x}) - v_m(\hat{x}) \leq u_m(\hat{x}) - g(\hat{x}) \leq 0$ which is already a contradiction. If $u_m(\hat{x}) \leq \mathcal{M}u_m(\hat{x})$, then select for $\varepsilon_2 > 0$ a $\hat{\zeta} = \hat{\zeta}_{\varepsilon_1,\varepsilon_2}$ such that $u_m(\Gamma(\hat{x}, \hat{\zeta})) + K(\hat{x}, \hat{\zeta}) + \varepsilon_2 > \mathcal{M}u_m(\hat{x})$. Then,

$$\begin{aligned} M - \varepsilon_1 < u_m(\hat{x}) - v_m(\hat{x}) &\leq u_m(\Gamma(\hat{x}, \hat{\zeta})) + K(\hat{x}, \hat{\zeta}) + \varepsilon_2 - \kappa/m - K(\hat{x}, \hat{\zeta}) - v_m(\Gamma(\hat{x}, \hat{\zeta})) \\ &\leq \varepsilon_2 - \kappa/m + M, \end{aligned}$$

which is a contradiction for $\varepsilon_1, \varepsilon_2$ sufficiently small. This shows that the supremum M can not be attained in $\mathbb{R}^d \setminus S$, neither can it be approached from within $\mathbb{R}^d \setminus S$.

Step 2. Now that we are sure we do not have to take into account the boundary conditions, we employ the doubling of variables device as usual. We define for $\varepsilon > 0$ and u_m, v_m chosen as in Lemma 2.5.10

$$M_\varepsilon = \sup_{x,y\in\mathbb{R}^d} \left(u_m(x) - v_m(y) - \frac{1}{2\varepsilon}|x - y|^2 \right).$$

In view of the definition of w and u_m, v_m, the maximum is attained in a compact set C (independent of small ε). Choose a point $(x_\varepsilon, y_\varepsilon) \in C$ where the maximum is attained. By applying Lemma 1.2.2, we obtain that $\frac{1}{2\varepsilon}|x_\varepsilon - y_\varepsilon|^2 \to 0$ as $\varepsilon \to 0$, and that $M_\varepsilon \to M = u_m(x_0) - v_m(x_0)$ for all limit points x_0 of (x_ε). We assume from now on wlog that we have chosen a convergent subsequence of (x_ε), (y_ε), converging to the same limit $x_0 \in C$. Let ε small enough such that $x_\varepsilon, y_\varepsilon \in S$ (by Step 1), and that all local estimates in (B1), (B2) hold.

Hence, we can apply Corollary 2.5.9 in $(x_\varepsilon, y_\varepsilon)$ for $\varphi(x, y) = \frac{1}{2\varepsilon}|x - y|^2$: For any $\delta > 0$, there is a range of $\alpha > 0$, for which there are matrices X, Y satisfying (2.43), and $(p, -q) = \nabla\varphi(x_\varepsilon, y_\varepsilon)$ (so $p = q = \frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon)$) such that

$$\begin{aligned} \min\Big(F(x_\varepsilon, u_m(x_\varepsilon), p, X, \mathcal{I}^{1,\delta}_\beta[x_\varepsilon, \varphi_\alpha(\cdot, y_\varepsilon)] + \mathcal{I}^{2,\delta}_\beta[x_\varepsilon, p, u_m(\cdot)]), u_m(x_\varepsilon) - \mathcal{M}u_m(x_\varepsilon)\Big) &\leq 0 \\ \min\Big(F(y_\varepsilon, v_m(y_\varepsilon), q, Y, \mathcal{I}^{1,\delta}_\beta[y_\varepsilon, -\varphi_\alpha(x_\varepsilon, \cdot)] + \mathcal{I}^{2,\delta}_\beta[y_\varepsilon, q, v_m(\cdot)]), v_m(y_\varepsilon) - \mathcal{M}v_m(y_\varepsilon)\Big) &\geq \frac{\kappa}{m}. \end{aligned}$$

Case 2a ($u_m(x_\varepsilon) - \mathcal{M}u_m(x_\varepsilon) \leq 0$): Using $v_m(y_\varepsilon) - \mathcal{M}v_m(y_\varepsilon) \geq \frac{\kappa}{m}$, for $\varepsilon > 0$ small enough,

$$\begin{aligned} M &= \limsup_{\varepsilon\to 0}(u_m(x_\varepsilon) - v_m(y_\varepsilon)) \\ &\leq \limsup_{\varepsilon\to 0} \mathcal{M}u_m(x_\varepsilon) - \liminf_{\varepsilon\to 0} \mathcal{M}v_m(y_\varepsilon) - \frac{\kappa}{m} \leq \mathcal{M}u_m(x_0) - \mathcal{M}v_m(x_0) - \frac{\kappa}{m}, \end{aligned}$$

where we have used the upper and lower semicontinuity of $\mathcal{M}u_m$ and $\mathcal{M}v_m$, respectively (Lemma 2.5.7). The contradiction is obtained as in Step 1.

Case 2b ($u_m(x_\varepsilon) - \mathcal{M}u_m(x_\varepsilon) > 0$): It remains to treat the PIDE part

$$F(x_\varepsilon, u_m(x_\varepsilon), p, X, \mathcal{I}^{1,\delta}_\beta[x_\varepsilon, \varphi_\alpha(\cdot, y_\varepsilon)] + \mathcal{I}^{2,\delta}_\beta[x_\varepsilon, p, u_m(\cdot)]) \leq 0 \tag{2.46}$$

$$F(y_\varepsilon, v_m(y_\varepsilon), q, Y, \mathcal{I}^{1,\delta}_\beta[y_\varepsilon, -\varphi_\alpha(x_\varepsilon, \cdot)] + \mathcal{I}^{2,\delta}_\beta[y_\varepsilon, q, v_m(\cdot)]) \geq \frac{\kappa}{m}. \tag{2.47}$$

Before we can proceed, we have to compare the integral terms in both inequalities. First note that because $|x + \ell(x, \beta, z) - y|^2 = |x - y|^2 + 2\langle x - y, \ell(x, \beta, z)\rangle + |\ell(x, \beta, z)|^2$, for all β

$$\mathcal{I}^{1,\delta}_\beta[x_\varepsilon, \varphi(\cdot, y_\varepsilon)] = \frac{1}{2\varepsilon}\int_{|z|<\delta} |\ell(x_\varepsilon, \beta, z)|^2 \nu(dz) < \infty$$

$$\mathcal{I}^{1,\delta}_\beta[y_\varepsilon, -\varphi(x_\varepsilon, \cdot)] = \frac{1}{2\varepsilon}\int_{|z|<\delta} -|\ell(y_\varepsilon, \beta, z)|^2 \nu(dz) < \infty,$$

(finite by (V4) and definition of $\mathcal{PB}$) so trivially $\mathcal{I}^{1,\delta}_\beta[x_\varepsilon, \varphi(\cdot, y_\varepsilon)] \leq \mathcal{I}^{1,\delta}_\beta[y_\varepsilon, -\varphi(x_\varepsilon, \cdot)] + \frac{1}{\varepsilon}o_\delta(1)$. Because we know that φ_α converges to φ uniformly in $C^2(C)$ for any compact C, we can see analogously to the proof of Corollary 2.5.9 that $\mathcal{I}^{1,\delta}_\beta[x_\varepsilon, \varphi_\alpha(\cdot, y_\varepsilon)] \leq \mathcal{I}^{1,\delta}_\beta[y_\varepsilon, -\varphi_\alpha(x_\varepsilon, \cdot)] + \frac{1}{\varepsilon}o_\delta(1) + o_\alpha(1)$, where $o_\alpha(1)$ may depend on ε, but is independent of small δ.

Using that $(x_\varepsilon, y_\varepsilon)$ is a maximum point and again $|x + y|^2 = |x|^2 + 2\langle x, y\rangle + |y|^2$,

$$u_m(x_\varepsilon + d) - u_m(x_\varepsilon) - \frac{1}{\varepsilon}\langle x_\varepsilon - y_\varepsilon, d\rangle \leq v_m(y_\varepsilon + d') - v_m(y_\varepsilon) - \frac{1}{\varepsilon}\langle x_\varepsilon - y_\varepsilon, d'\rangle + \frac{1}{2\varepsilon}|d - d'|^2, \tag{2.48}$$

where d, d' are arbitrary vectors. We find by integrating (2.48) for all β and $d = \ell(x_\varepsilon, \beta, z)$, $d' = \ell(y_\varepsilon, \beta, z)$ that

$$\begin{aligned}\mathcal{I}^{2,\delta}_\beta[x_\varepsilon, p, u_m(\cdot)] \leq \mathcal{I}^{2,\delta}_\beta[y_\varepsilon, q, v_m(\cdot)] &+ \frac{1}{2\varepsilon}\int_{|z|\geq\delta} |\ell(x_\varepsilon, \beta, z) - \ell(y_\varepsilon, \beta, z)|^2\, \nu(dz) \\ &+ \int_{|z|\geq 1} \langle p, \ell(x_\varepsilon, \beta, z) - \ell(y_\varepsilon, \beta, z)\rangle\, \nu(dz).\end{aligned}$$

We then have by (B1) for $\varepsilon > 0$ small enough, (denoting $l^1_\beta = \mathcal{I}^{1,\delta}_\beta[x_\varepsilon, \varphi(\cdot, y_\varepsilon)] + \mathcal{I}^{2,\delta}_\beta[x_\varepsilon, p, u_m(\cdot)]$ and $l^2_\beta = \mathcal{I}^{1,\delta}_\beta[y_\varepsilon, -\varphi(x_\varepsilon, \cdot)] + \mathcal{I}^{2,\delta}_\beta[y_\varepsilon, q, v_m(\cdot)]$) that

$$l^1_\beta \leq l^2_\beta + O(\frac{1}{\varepsilon}|x_\varepsilon - y_\varepsilon|^2) + \frac{1}{\varepsilon}o_\delta(1) + o_\alpha(1),$$

where $O(\frac{1}{\varepsilon}|x_\varepsilon - y_\varepsilon|^2)$, $o_\delta(1)$ and $o_\alpha(1)$ are independent of β because of (B1). Likewise, $O(\frac{1}{\varepsilon}|x_\varepsilon - y_\varepsilon|^2)$ is independent of δ and α. Thus

$$\begin{aligned}\frac{\kappa}{m} &\leq F(y_\varepsilon, v_m(y_\varepsilon), q, Y, l^2_\beta) - F(x_\varepsilon, u_m(x_\varepsilon), p, X, l^1_\beta) && \text{by (2.46) and (2.47)} \\ &\leq F(y_\varepsilon, v_m(y_\varepsilon), q, Y, l^2_\beta) - F(x_\varepsilon, v_m(y_\varepsilon), p, X, l^1_\beta) && \text{for small } \varepsilon \text{ because } c \geq 0 \\ &\leq F(y_\varepsilon, v_m(y_\varepsilon), q, Y, l^1_\beta) - F(x_\varepsilon, v_m(y_\varepsilon), p, X, l^1_\beta) + O(\frac{1}{\varepsilon}|x_\varepsilon - y_\varepsilon|^2) + \frac{1}{\varepsilon}o_\delta(1) + o_\alpha(1),\end{aligned}$$

where we have used ellipticity (P1) and Lipschitz continuity (P3) in the last component for the O and o values independent of β. The matrix inequality (2.43) becomes

$$-\frac{1}{\alpha}I \leq \begin{bmatrix} X & 0 \\ 0 & -Y \end{bmatrix} \leq \frac{1}{\varepsilon}\begin{bmatrix} I & -I \\ -I & I \end{bmatrix} + o_\alpha(1). \tag{2.49}$$

By assumption (B2) for $R > 0$ large enough (v_m is locally bounded) and ε small enough,

$$\frac{\kappa}{m} \leq \omega_R(|x_\varepsilon - y_\varepsilon| + \varepsilon^{-1}|x_\varepsilon - y_\varepsilon|^2) + O(\frac{1}{\varepsilon}|x_\varepsilon - y_\varepsilon|^2) + \frac{1}{\varepsilon}o_\delta(1) + o_\alpha(1).$$

Now let subsequently converge $\delta \to 0$ (because of the special dependence of α – the smaller δ, the larger α – this does not affect α), and then $\alpha \to 0$.[7] The contradiction is finally obtained by $\varepsilon \to 0$. □

Remark 2.5.7. Our notion of viscosity solutions used here is different from the concept of discontinuous viscosity solutions (as exposed in §1.3) in one respect: We do not consider the upper and lower semicontinuous envelopes of G on the boundary, where

$$G(x, r, p, X, u(\cdot)) = \begin{cases} \min(F(x, r, p, X, u(\cdot)), r - \mathcal{M}u(\cdot)) & x \in S \\ \min(r - g(x), r - \mathcal{M}u(\cdot)) & x \notin S. \end{cases}$$

To get uniqueness of discontinuous viscosity solutions on $\overline{S}$ (or $\mathbb{R}^d$), a *strong comparison* result has to be proved, which typically involves finding conditions for continuity at the boundary; see Barles and Rouy [13] in the case of HJB equations, and Barles et al. [15] for PIDEs. By requiring continuity of v at the boundary already for the existence proof, we have avoided such a tedious procedure in the comparison proof.

2.5.5 Parabolic case

Now let us deduce the parabolic result from the preceding discussion. We will keep the presentation short, only outlining the differences to the elliptic case. We recall the form of the parabolic HJBQVI (where $\partial^+ S_T$ denotes the parabolic nonlocal boundary):

$$\begin{aligned} \min(-\sup_{\beta \in B}\{u_t + \mathcal{L}^\beta u + f^\beta\}, u - \mathcal{M}u) &= 0 \qquad \text{in } S_T \\ \min(u - g, u - \mathcal{M}u) &= 0 \qquad \text{in } \partial^+ S_T, \end{aligned} \tag{2.10}$$

where for $y = (t, x)$

$$\begin{aligned} \mathcal{L}^\beta u(y) = \frac{1}{2} tr\left(\sigma(y, \beta)\sigma^T(y, \beta) D^2 u(y)\right) + \langle \mu(y, \beta), \nabla u(y) \rangle \\ + \int u(t, x + \ell(y, \beta, z)) - u(y) - \langle \nabla u(y), \ell(y, \beta, z) \rangle 1_{|z|<1}\, \nu(dz). \end{aligned} \tag{2.11}$$

The function F then is

$$F(x, r, p, X, (l_\beta)) = -\sup_{\beta \in B} \left\{ \frac{1}{2} tr\left(\sigma(x, \beta)\sigma^T(x, \beta) X\right) + \langle \mu(x, \beta), p \rangle + f(x, \beta) + l_\beta \right\},$$

and the HJBQVI (with the obvious adjustments, compare also §2.4.1) reads

$$\min(-u_t(y) + F(y, u(y), \nabla_x u(y), D_x^2 u(y), \mathcal{I}_\beta[y, u(t, \cdot)]), u(y) - \mathcal{M}u(y)) = 0.$$

All assumptions and the definition of the space $\mathcal{PB} = \mathcal{PB}([0, T) \times \mathbb{R}^d)$ are as introduced in §2.2. The test functions φ are now in $C^{1,2}([0, T) \times \mathbb{R}^d)$ (once continuously differentiable in time).

[7] By $\alpha \to 0$, we lose the first part of the inequality (2.49) (so we can not be sure anymore that X, Y are bounded because they are dependent on α).

Instead of $U_\delta(t_0,x_0) \subset \mathbb{R}^d$, we consider $[0,T) \times U_\delta(t_0,x_0)$, where the $[0,T)$ could in principle be any time interval open in $[0,T)$ containing t_0.
Recall the definition of a viscosity solution of (2.10) from §2.4.1. The original motivation for introducing the different definitions of viscosity solutions was to cater for the singularity in the integral. Because this integral, started in (t_0,x_0), only takes into account values at the time t_0, the different definitions in §2.5.2 are equivalent in the parabolic case, too (where the time derivative in $t=0$ is only the one-sided derivative).

For the third definition (Definition 2.5.5), we need the parabolic semijets $P^+u(t,x)$ and $P^-u(t,x)$ on $[0,T) \times \mathbb{R}^d$,

$$\begin{aligned} P^+u(t,x) = \{(a,p,X) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d : u(t+s,x+z) \le u(t,x) + as + \langle p,z\rangle \\ + \frac{1}{2}\langle Xz,z\rangle + o(|s|+|z|^2) \text{ as } s,z \to 0,\ (t+s,z) \in [0,T) \times \mathbb{R}^d\}, \end{aligned}$$

and P^-u analogously. $\overline{P}^+u$ and $\overline{P}^-u$ denote the corresponding limiting parabolic semijets. The reformulation of Definition 2.5.5 is then "*For* $(a,p,X) \in P^+u^*(t_0,x_0)$ *with* $p = D_x\varphi(t_0,x_0)$ *and* $X \le D_x^2\varphi(t_0,x_0)$*, ...*", and in the same way for the supersolution part (we have requirements only on p and X because only they need to be consistent with φ as used in the $I^{1,\delta}$ integral).

Finally, we obtain the parabolic maximum principle for impulse control:

Corollary 2.5.13. *Assume (V1), (V2). Let u be a viscosity subsolution and v a viscosity supersolution of (2.10), and $\varphi \in C^{1,2}([0,T) \times \mathbb{R}^{2d})$. If $(t_0,x_0,y_0) \in \mathbb{R}^{2d+1}$ is a global maximum point of $u^*(t,x) - v_*(t,y) - \varphi(t,x,y)$ on $[0,T) \times \mathbb{R}^d$, then, for any $\delta > 0$, there exists $\bar\alpha$ such that for $0 < \alpha < \bar\alpha$, there are $(a,p,X) \in \overline{P}^+u^*(t_0,x_0)$ and $(b,q,Y) \in \overline{P}^-v_*(t_0,y_0)$ with*

$$\begin{aligned} \min\Big(-a + F(x_0,u^*(x_0),p,X,\mathcal{I}_\beta^{1,\delta}[x_0,\varphi_\alpha(\cdot,y_0)] + \mathcal{I}_\beta^{2,\delta}[x_0,p,u^*(\cdot)]),u^* - \mathcal{M}u^*\Big) &\le 0 \\ \min\Big(-b + F(y_0,v_*(y_0),q,Y,\mathcal{I}_\beta^{1,\delta}[y_0,-\varphi_\alpha(x_0,\cdot)] + \mathcal{I}_\beta^{2,\delta}[y_0,q,v_*(\cdot)]),v_* - \mathcal{M}v_*\Big) &\ge 0 \end{aligned}$$

if $x_0 \in S$ or $y_0 \in S$, respectively. Here $a+b = \varphi_t(t_0,x_0,y_0)$, $p = \nabla_x\varphi(t_0,x_0,y_0) = \nabla_x\varphi_\alpha(t_0,x_0,y_0)$, $q = -\nabla_y\varphi(t_0,x_0,y_0) = -\nabla_y\varphi_\alpha(t_0,x_0,y_0)$, and furthermore,

$$-\frac{1}{\alpha}I \le \begin{bmatrix} X & 0 \\ 0 & -Y \end{bmatrix} \le D^2_{(x,y)}\varphi_\alpha(t_0,x_0,y_0) = D^2_{(x,y)}\varphi(t_0,x_0,y_0) + o_\alpha(1). \tag{2.50}$$

The proof is in no way different, once the parabolic Jensen-Ishii lemma, proved by the same technique as Lemma 2.5.8, is available. The requirement $a+b = \varphi_t(t_0,x_0,y_0)$ is immediately plausible from the necessary first order criterion in time. Compare also Barles and Imbert [11], Crandall et al. [33].

Finally, a comparison theorem can be formulated. Lemma 2.5.10 is still true in the parabolic case, due to the linearity of the differential operator.

Define $\mathcal{PB}_p = \mathcal{PB}_p([0,T] \times \mathbb{R}^d)$ in the parabolic case by all functions $u \in \mathcal{PB}$, for which there is a (time-independent!) constant C such that $|u(t,x)| \le C(1+|x|^p)$ for all $(t,x) \in [0,T] \times \mathbb{R}^d$. The upper (lower) semicontinuous envelope u^* (v_*) is again taken from within $[0,T] \times \mathbb{R}^d$.

Theorem 2.5.14. *Let Assumptions 2.2.1, 2.2.3 and 2.2.4 be satisfied. Assume further that there is a $w \ge 0$ as in Lemma 2.5.10 (for a constant $\kappa > 0$) with $w(t,x)/|x|^p \to \infty$ for $|x| \to \infty$ (uniformly in t). If $u \in \mathcal{PB}_p([0,T] \times \mathbb{R}^d)$ is a subsolution and $v \in \mathcal{PB}_p([0,T] \times \mathbb{R}^d)$ a supersolution of (2.10), then $u^* \le v_*$ on $[0,T] \times \mathbb{R}^d$.*

Corollary 2.5.15. *Under the same assumptions, there is at most one viscosity solution in $\mathcal{PB}_p([0,T]\times\mathbb{R}^d)$ of (2.10), and it is continuous on $[0,T]\times\mathbb{R}^d$.*

Proof of Theorem 2.5.14: We only point out the differences to the elliptic Theorem 2.5.11. Define $M := \sup_{t\in[0,T],x\in\mathbb{R}^d} u_m(t,x) - v_m(t,x)$ and assume it is > 0. Step 1 is proved as in the elliptic case, but on the parabolic boundary $\partial^+ S_T$. For Step 2, we again know that the supremum

$$M_\varepsilon = \sup_{t\in[0,T],x,y\in\mathbb{R}^d}\left(u_m(t,x) - v_m(t,y) - \frac{1}{2\varepsilon}|x-y|^2\right)$$

is attained in a compact set of $[0,T]\times\mathbb{R}^d\times\mathbb{R}^d$ (independent of small ε), say in $(t_\varepsilon, x_\varepsilon, y_\varepsilon)$. For small enough ε, we know by Step 1 that $t_\varepsilon < T$ and $x_\varepsilon, y_\varepsilon \in S$. We proceed as in the elliptic case, and arrive at the PIDE case 2b. All integral estimates hold because t_ε is fixed at the moment, and the conclusion is exactly the same (the time derivatives a and b cancel out when subtracting the PIDE sub/supersolution inequalities). The modulus of continuity needs to exist locally uniformly in t_ε before letting ε converge to 0. $\square$

2.6 Extension to state-dependent intensity

We will extend in this section the results obtained so far in this chapter to the case of a state-dependent intensity.

A process with state- or time-dependent intensity can be represented in the framework of SDE (2.4) by setting the intensity measure $\nu(dz) = 1_{[0,\infty)}Leb(dz)$ and ℓ equal to an indicator function: for example, the process $\int_0^t\int_{\mathbb{R}} 1_{z\le\lambda_s}\, N(dz,ds)$ has a time-dependent intensity λ_s (and the jump measure effectively has bounded support $[0,\lambda_s]$).

Obviously, the continuity of ℓ in (V3) does not hold; however, if the intensity depends continuously on time and state, then the Hamiltonian $\sup_{\beta\in B} u_t + \mathcal{L}^\beta u + f^\beta$ is continuous for $u \in \mathcal{PB}\cap C^{1,2}([0,T)\times\mathbb{R}^d)$. As this is the only consequence of (V3) used in the existence proof, the existence result Theorem 2.4.2 adapts readily to a time-dependent intensity.

But still, (V3) is needed in the course of the uniqueness proof. What is more, condition (B1) is in general not satisfied for a state-dependent intensity: Because the square of an indicator function is still the same indicator function, only the inequality

$$\int |\ell(x,\beta,z) - \ell(y,\beta,z)|^2\nu(dz) \le C|x-y|$$

holds. Yet the results about viscosity solution uniqueness of the HJBQVIs (2.15), (2.10) still hold under certain assumptions. We will in the following stick to the elliptic setting of the HJBQVI as in §2.5, with the assumptions formulated in time-dependent version. The corresponding parabolic results hold under these conditions in exactly the same way.

The integral $\mathcal{I}$ in (2.15) is assumed to consist of two parts:

$$\mathcal{I}_\beta[x,\phi] = \int \phi(x+\ell^1(x,\beta,z)) - \phi(x) - \langle\nabla\phi(x), \ell^1(x,\beta,z)\rangle 1_{|z|<1}\,\nu^1(dz) + \mathcal{J}_\beta[x,\phi]$$

$$\mathcal{J}_\beta[x,\phi] = \int \phi(x+\ell^2(x,\beta,z)) - \phi(x) - \langle\nabla\phi(x), \ell^2(x,\beta,z)\rangle 1_{|z|<1}\,\nu^2(dz),$$

where the ν^2 part is meant to include the state-dependent intensity.

Assumption 2.6.1. *(BS1) Assume (B2), and (B1) for ν^1, ℓ^1.*

(BS2) For three sequences $(x_n)_{n\geq 1}, (y_n)_{n\geq 1} \subset S$, $(t_n)_{n\geq 1} \subset [0,T)$, define $A_{n,\beta} := \{z \in \mathbb{R}^{k_2} : \ell^2(t_n, x_n, \beta, z) \neq \ell^2(t_n, y_n, \beta, z)\}$. Assume that if $(x_n), (y_n)$ converge to the same limit in S and $t_n \to t \in [0,T)$, then

$$\nu^2 \left(\bigcup_{\beta \in B} A_{n,\beta} \right) \longrightarrow 0$$

for $n \to \infty$, and $\ell^2(t_n, x_n, \beta, A_{n,\beta})$, $\ell^2(t_n, y_n, \beta, A_{n,\beta})$ are bounded independent of $\beta \in B$ for large n.

Under assumption (BS2), Prop. 2.5.1 can be shown to hold for ν^2, ℓ^2 (similar to proof below). The different definitions are equivalent, as the $I^{1,\delta}$ part for ν^2, ℓ^2 becomes redundant. Corollary 2.5.9 uses only Prop. 2.5.1 for integrals with finite measure.

Theorem 2.6.1. *Let Assumptions 2.2.1 (for ℓ^1), 2.2.3 and 2.6.1 be satisfied and c be locally Lipschitz continuous. Assume further that there is a $w \geq 0$ as in Lemma 2.5.10 (for a constant $\kappa > 0$) with $w(x)/|x|^p \to \infty$ for $|x| \to \infty$. If $u \in \mathcal{PB}_p(\mathbb{R}^d)$ is a subsolution and $v \in \mathcal{PB}_p(\mathbb{R}^d)$ a supersolution of (2.15), then $u^* \leq v_*$.*

Corollary 2.6.2 (Viscosity solution: uniqueness)**.** *Under the same assumptions, there is at most one viscosity solution in $\mathcal{PB}_p(\mathbb{R}^d)$ of (2.15), and it is continuous.*

Proof of Theorem 2.6.1: The proof is exactly the same as that of Theorem 2.5.11, except when comparing the integrals. We compare the ν^1 part of $\mathcal{I}$ as in that theorem (by (BS1)).

We recall that we have wlog sequences $(x_\varepsilon)_\varepsilon$ and $(y_\varepsilon)_\varepsilon$ converging to the same limit for $\varepsilon \to 0$ such that $(x_\varepsilon, y_\varepsilon)$ is a maximum point of $u_m(x) - v_m(y) - \frac{1}{2\varepsilon}|x-y|^2$. Define $A_{\varepsilon,\beta}$ as in Assumption 2.6.1. Then for $z \notin A_{\varepsilon,\beta}$,

$$u_m(x_\varepsilon + \ell(x_\varepsilon, \beta, z)) - u_m(x_\varepsilon) \leq v_m(y_\varepsilon + \ell(y_\varepsilon, \beta, z)) - v_m(y_\varepsilon). \tag{2.51}$$

Subtracting $\langle p, \ell(x_\varepsilon, \beta, z)\rangle = \frac{1}{\varepsilon}\langle x_\varepsilon - y_\varepsilon, \ell(x_\varepsilon, \beta, z)\rangle$ from both sides of (2.51) for $|z| < 1$ and integrating, we get

$$\begin{aligned}
&\mathcal{J}^{2,\delta}_\beta[x_\varepsilon, p, u_m(\cdot)] \leq \mathcal{J}^{2,\delta}_\beta[y_\varepsilon, q, v_m(\cdot)] \\
&\quad + \nu^2(A_{\varepsilon,\beta}) \left(\sup_{z \in A_{\varepsilon,\beta}} |u_m(x_\varepsilon + \ell(x_\varepsilon, \beta, z)) - u_m(x_\varepsilon) - \langle p, \ell^2(x_\varepsilon, \beta, z)\rangle 1_{|z|<1}| \right. \\
&\quad \left. + \sup_{z \in A_{\varepsilon,\beta}} |v_m(y_\varepsilon + \ell(y_\varepsilon, \beta, z)) - v_m(y_\varepsilon) - \langle p, \ell^2(y_\varepsilon, \beta, z)\rangle 1_{|z|<1}| \right).
\end{aligned}$$

By (BS2), the term on the right converges to 0 for $\varepsilon \to 0$, independently of β, δ. The rest of the proof proceeds exactly as in Theorem 2.5.11. □

Example 2.6.1. Consider a jump-diffusion process evolving according to $dX_t = dW_t + dN_t$, where W is standard one-dimensional Brownian motion, and N is a Poisson process whose intensity $\lambda(X_{t-})$ depends continously on the current state. In this case $\nu^2(dz) = 1_{[0,\infty)} Leb(dz)$ and $\ell^2(x,z) = 1_{z \leq \lambda(x)}$, and ν^2 has effectively bounded support. If $x_n \to x$, $y_n \to x$, then $\nu^2(A_n) \to 0$.

Chapter 3

Numerical schemes for (HJB)QVIs and iterated optimal stopping

This chapter serves as an introduction to numerical schemes for quasi-variational inequalities. Our mathematical investigation focuses on approximation by the so-called iterated optimal stopping technique.

The plan of the chapter is as follows: After an introduction to iterated optimal stopping and the contents of this chapter in §3.1, we present a survey of different numerical methods for QVIs. In §3.3, we establish viscosity solution existence and uniqueness results of the HJB variational inequality (HJBVI) of optimal stopping. We introduce in §3.4 two equivalent optimal stopping iterations, and prove the convergence of iterated optimal stopping using a viscosity solution stability result. In passing by, we prove an HJBQVI existence and uniqueness result under slightly weaker assumptions than in Chapter 2. The last section §3.5 gives a short account of the actual numerical implementation of iterated optimal stopping.

3.1 Introduction

We consider in this chapter the numerical solution of HJBQVIs, which read in the parabolic case:

$$\begin{aligned} \min(-\sup_{\beta\in B}\{u_t + \mathcal{L}^\beta u + f^\beta\}, u - \mathcal{M}u) &= 0 \qquad \text{in } S_T \\ \min(u - g, u - \mathcal{M}u) &= 0 \qquad \text{in } \partial^+ S_T, \end{aligned} \tag{3.1}$$

with $\mathcal{L}^\beta$ from (2.11) the infinitesimal generator of the process X, and the intervention operator $\mathcal{M}$ from (2.9) selecting the optimal instantaneous impulse. The basic problem preventing a straightforward numerical solution of (3.1) is that the equation we want to solve already refers to the (still unknown) solution u via the nonlocal intervention operator $\mathcal{M}u$.

Iterated optimal stopping. If one restricts the number of allowed impulses to $n \in \mathbb{N}_0$, then it is possible to find an optimal impulse strategy by solving a sequence of optimal stopping problems. In PDE terms, we approximate the solution of the quasi-variational inequality (3.1) by recursively solving a sequence of variational inequalities. The first step for $n = 0$ does not include any impulses; it is represented by the solution v^0 of

$$\begin{aligned} -\sup_{\beta\in B}\{u_t + \mathcal{L}^\beta u + f^\beta\} = 0 \qquad & \text{in } S_T \\ u - g = 0 \qquad & \text{in } \partial^+ S_T. \end{aligned} \tag{3.2}$$

For $n \geq 1$ we define v^n to be the solution of the variational inequality[1]

$$\begin{aligned} \min(-\sup_{\beta\in B}\{u_t + \mathcal{L}^\beta u + f^\beta\}, u - \mathcal{M}v^{n-1}) = 0 \qquad & \text{in } S_T \\ \min(u - g, u - \mathcal{M}v^{n-1}) = 0 \qquad & \text{in } \partial^+ S_T, \end{aligned} \tag{3.3}$$

which is the PIDE corresponding to an optimal stopping problem with stopping payoff $\mathcal{M}v^{n-1}$ except on $\partial^+ S_T$, where the payoff is $\max(g, \mathcal{M}v^{n-1})$; see §3.4 for details. The iterate v^1 contains one optimally placed impulse, the iterate v^2 includes two impulses, and so on. An optimal impulse strategy can be derived from the optimal stopping rule in (3.3): Jump when $v^n = \mathcal{M}v^{n-1}$ for the first time, and for the last time when $v^1 = \mathcal{M}v^0$.

We will investigate closer iterated optimal stopping in the subsequent sections mainly because of the following two reasons:

(i) Each iteration step n bears an intuitive stochastic interpretation as impulse control problem with at most n impulses; the convergence speed is relatively easy to obtain, and depends on well known quantities.

(ii) Optimal stopping problems can be solved efficiently: There is a relatively long experience with such problems, and an extensive literature on different (PDE and Monte Carlo) methods to solve optimal stopping problems for American options.

The **main objective** of this chapter is to prove the convergence of the iterated optimal stopping value functions v^n or solutions of (3.2), (3.3) to the value function of stochastic and impulse control in the same general setting as in Chapter 2, by combining stochastic and viscosity solution techniques. Our result is to our knowledge the first one for jump-diffusions using viscosity solution techniques. Furthermore, we establish results for the exit time problem on possibly finite time horizon, which seems to be a very unusual setting in the literature; of the papers surveyed below, all except [17] and [79] consider only the infinite horizon, mostly on a domain which is never left (no exit time).

Iterated optimal stopping goes back to the early days of the analysis of impulse control problems and QVIs; see, e.g., Bensoussan and Lions [17]. A common approach is to first establish the correspondence between variational inequalities (VIs) and optimal stopping, and then to prove existence of a QVI solution as limit of solutions of suitable VIs. Indeed, as already mentioned in §2.1, many existence and uniqueness results for QVIs make use of iterated optimal stopping, and thus contain implicitly a convergence proof for this iteration.

So there are two routes to prove the convergence of iterated optimal stopping: either investigate the convergence of solutions of (3.3), or analyze directly the value functions of the optimal stopping problems corresponding to (3.3).

The first route is taken by, e.g., Bensoussan and Lions [17] and Baccarin and Sanfelici [7] for solutions in Sobolev spaces, and in Menaldi [82], [83] using a variational approach. All these works

[1] The prefix *quasi* is omitted in the name because here the obstacle part $\mathcal{M}v^{n-1}$ does not refer to the solution anymore.

deal with pure impulse control of diffusions, except [83] which is in a jump-diffusion setting. In the context of viscosity solutions, Lenhart [76] for piecewise deterministic processes and Morimoto [86] for a particular controlled diffusion are the only references known to the author. While [76] exclusively uses viscosity solution techniques to prove convergence, [86] employs also stochastic techniques.

As for the second route, Davis [37] (for piecewise deterministic processes) and Palczewski and Zabczyk [95] (for jump-diffusions) directly prove that the value functions v^n converge to the fixed point of a dynamic programming equation in spaces of continuous functions. Convergence proofs by purely stochastic means for diffusions can be found for specific portfolio optimization problems in Chancelier et al. [27] (with stochastic control) and Ly Vath and Mnif [79]; see also Øksendal and Sulem [93] for a generalization of [27] to jump-diffusions.

Overview of the chapter. We briefly sketch in the following the main steps of our proof of convergence. First of all, we have to establish that the value function of combined stochastic control and stopping is the unique viscosity solution of an HJBVI. We deduce this result with only marginal extra work from the corresponding HJBQVI result of Chapter 2, in the same general setting and under similar assumptions. This result is new in its generality; compare also the introductory discussion in Chapter 2, which applies here, too. Viscosity solution existence and uniqueness for (HJB)VIs has previously been proved for diffusions in Øksendal and Reikvam [91], Ceci and Bassan [26]; see also Bensoussan and Lions [16], [17] for solutions in Sobolev spaces. Pham [99] analyzes the problem for combined stochastic control and stopping of a jump-diffusion on a finite horizon, but in a less general setting than ours without exit time.

A main requirement of our HJBVI existence and uniqueness result is that the obstacle be continuous. Now the iteration (3.3) has a discontinuous obstacle: it is $\mathcal{M}v^{n-1}$ in S_T, and $\max(g, \mathcal{M}v^{n-1})$ on $\partial^+ S_T$. To circumvent this problem, we introduce a different optimal stopping iteration with obstacle $\max(v^{n-1}, \mathcal{M}v^{n-1})$, which is continuous if the previous iterate v^{n-1} is continuous (under suitable assumptions on $\mathcal{M}$). We show that this new optimal stopping iteration is equivalent to the old one, and thus the solution of (3.3) is continuous. (Note that the continuity can also be obtained by different means, see Remark 3.4.1.)

This interesting new equivalence intuitively means that if less than n impulses are optimal, then in iteration n the controller can decide to let go of the first impulse, and continue with $n-1$ remaining possible impulses.

For the proof of convergence, we use a technique that consists in establishing contraction-like properties of the operator mapping v^{n-1} to the solution of (3.3); this technique is commonly used in the analysis of QVIs and seems to be due to Hanouzet and Joly [60] (see also the references in Lenhart [76]). Our use of the technique carves out the crucial role that the HJBQVI strict supersolution w from Chapter 2 plays for the convergence. The stability of HJBVI viscosity solutions with respect to the obstacle (Theorem 3.3.8) and the continuity of v^n permit us to prove the convergence to the HJBQVI viscosity solution. As a little compensation for the extra work, we obtain an HJBQVI existence and uniqueness result under assumptions slightly weaker than in Chapter 2, just in the spirit of the original use of iterated optimal stopping.

The chapter is organized as follows: We start with a survey of alternative numerical methods to solve QVIs in §3.2.The following §3.3 investigates the general stochastic control and optimal stopping problem, and relates its value function to the viscosity solution of an HJBVI. After proving the stability of HJBVI viscosity solutions with respect to the obstacle, we come to the detailed exposition of iterated optimal stopping in §3.4. By the equivalence of the two optimal stopping iterations, we can conclude that the solution of (3.3) is continuous. We establish the convergence result for the parabolic case (Theorem 3.4.6), and deduce the corresponding result for the elliptic, time-independent case. In the last section (§3.5), we present a possible numerical implementation of iterated optimal stopping, and analyze convergence of the discrete approximation using viscosity solution techniques.

3.2 Alternative methods for solving (HJB)QVIs

We survey in this section alternative methods for the solution of HJBQVIs without iterated optimal stopping. Together with the parabolic HJBQVI (3.1), we consider the solution of the elliptic HJBQVI

$$\begin{aligned} \min(-\sup_{\beta\in B}\{-\rho u + \mathcal{L}^\beta u + f^\beta\}, u - \mathcal{M}u) = 0 \qquad & \text{in } S \\ \min(u - g, u - \mathcal{M}u) = 0 \qquad & \text{in } \mathbb{R}^d \setminus S. \end{aligned} \tag{3.4}$$

For the informal introduction below, we mainly confine ourselves to the QVI part of (3.1) and (3.4) corresponding to impulse control, i.e., the case $B = \{\beta_0\}$.

From the numerical point of view, there is quite some difference between the solution of the parabolic HJBQVI (3.1) and the elliptic version (3.4). The parabolic PIDE is a time-dependent equation, which means that typically we are given a terminal condition which can serve as proxy for the solution u at the previous time step; in this way one can hope that a backward solution yields a reasonably exact solution for small time stepping. The same is not true for the elliptic PIDE (3.4), where the equation is in general completely implicit. Nonetheless, much that can be said applies to both (3.1) and (3.4); therefore frequently, when referring to only one of the two, we actually mean both.

We formulate the algorithms below in discretized or non-discretized versions of (3.1), depending on which one suits better. Discretization is always the first step in a standard numerical solution of a PIDE, where the derivatives are replaced by their finite difference approximations, and the integrals are approximated by some quadrature rule. For an introduction to numerical methods for PDEs, we refer to Quarteroni et al. [102], Hundsdorfer and Verwer [62], or [108].

Standard methods

First we present two further standard methods, which seem to be well established and / or are frequently used.

Policy iteration. We present the policy iteration algorithm (see Chancelier et al. [28], Øksendal and Sulem [93]) for elliptic problems. It solves the fixed-point problem

$$u = \max(\sup_{\beta\in B} \mathcal{L}_\delta^\beta u, \sup_{\zeta\in Z_\delta} M^\zeta u), \tag{3.5}$$

which is a reformulated HJBQVI discretized with step δ, and $M^\zeta u(x) = u(\Gamma(x,\zeta)) + K(x,\zeta)$ the intervention operator for fixed ζ. Using the (discrete) intervention region R as additional variable, (3.5) becomes

$$u = \sup_{R,\beta,\zeta} \mathcal{O}_{R,\beta,\zeta} u, \tag{3.6}$$

where $\mathcal{O}$ is a suitable operator chosen to match (3.5). Each iteration then consists of two steps:

1. Given an iterate u^k, compute the maximal $\hat{\beta}$ and $\hat{\zeta}$ for $\mathcal{L}^{\beta}_{\delta}u^k$ and $M^{\zeta}u^k$, and the ensuing set $\hat{R}$.

2. Solve the fixed point problem $u = \mathcal{O}_{\hat{R},\hat{\beta},\hat{\zeta}}u$ and call the resulting solution u^{k+1}.

This fixed-point iteration converges because the HJB part is a contraction (because of discounting or finite time horizon), and the second impulse part is non-expansive, as proved in Chancelier et al. [28].

Backward solution (only parabolic). This direct method (see Chen and Forsyth [29]) proceeds by a backward induction, starting with the terminal condition of (3.1). For each time step j, the following steps are performed to obtain the approximate discretized solution u^j at this timestep:

1. Do implicit or explicit PDE backward timestepping for the HJB part of (3.1) to obtain $\tilde{u}^j$ from u^{j+1}.

2. Apply the intervention operator $\mathcal{M}$ to $\tilde{u}^j$ and set $u^j := \max(\tilde{u}^j, \mathcal{M}\tilde{u}^j)$.

Typically, the condition $\min(-\sup_{\beta\in B}\{u_t + \mathcal{L}^{\beta}u + f^{\beta}\}, u - \mathcal{M}u) = 0$ (or its discretized form) is not satisfied exactly, but will be a good approximation especially if the time grid spacing is very small. This plain backward solution is sometimes also called *dynamic programming*, and is known for American options as Brennan-Schwartz algorithm.

Methods for high-dimensional problems

We now come to non-standard methods, which are suited for high-dimensional problems.

A key problem in numerical analysis is the so-called "curse of dimensionality", which roughly means that the number of grid points needed to find an approximate solution of a PDE (or an integral) grows exponentially in the dimension. For a standard rectangular grid with 100 grid points in each direction, the total number of grid points in $\mathbb{R}^d$ is 100^d, which is barely manageable for $d \geq 4$.

This curse of dimensionality is already a problem for normal local PDEs. For QVIs, an additional challenge is that for the calculation of $\mathcal{M}u$, the approximate solution u may be needed on the full grid. If the set of impulse arrival points $\Gamma(t, x, Z(t, x))$ is k-dimensional, then in general the dimension of the discretization can at most be reduced from d to k.

Asymptotic expansion. If the fixed costs of the impulse control problem are relatively small, then asymptotic expansion may be the method of choice (see Atkinson et al. [6], Korn [71] in portfolio optimization context). It makes use of a known (analytical) solution of an HJB equation corresponding to a problem without transaction costs, and performs an expansion for small fixed costs. [6] apply their method to portfolio optimization examples for up to $d = 30$.

Monte Carlo methods. Instead of using a finite difference scheme, the methods presented above can also be combined with a Monte Carlo simulation, at least for parabolic problems. A Monte Carlo method has the advantage that no grid and no boundary conditions are needed. To give an example, iterated optimal stopping can use one of the many well developed Monte Carlo methods for the solution of optimal stopping problems; a standard technique for computing the conditional expectations in an optimal stopping problem is the Longstaff-Schwartz approach (see, e.g., Glasserman [55]). In a concrete portfolio optimization problem, such a Monte Carlo iterated optimal stopping method was used in Ly Vath and Mnif [79]. A Monte Carlo variant of the backward solution method was employed in Carmona and Ludkovski [24] for an optimal switching problem on a finite horizon.

Penalty methods involving BSDEs. Another recent development is a method involving penalized backward SDEs (see Kharroubi et al. [70], Elie and Kharroubi [43]). The viscosity solution v of the parabolic QVI for a diffusion X is associated with the minimal solution $Y_t = v(t, X_t)$ of a constrained BSDE. This Y can be approximated by the solution triple (Y^n, Z^n, U^n) of the following penalized BSDE:

$$\begin{aligned} Y_t^n = g(X_T) + \int_t^T f(X_s)ds + n\int_t^T \int_E (U_s^n(\zeta))^+\lambda(d\zeta)ds \\ - \int_t^T Z_s^n dW_s - \int_t^T \int_E (U_s^n(\zeta) - K(X_{s-}, \zeta))\mu(d\zeta, ds), \end{aligned} \tag{3.7}$$

where μ is a Poisson random measure with finite intensity measure λ supported on a compact set E; see [70] for details. Either the BSDE (3.7) can be solved numerically by using a regression method to estimate the arising conditional expectations, or its penalized PDE counterpart can be solved:

$$-u_t - \mathcal{L}u - f - n\int_E (M^\zeta u - u)^+\lambda(de) = 0, \qquad \text{in } S_T \tag{3.8}$$

with $M^\zeta u(t,x) = u(t, \Gamma(t,x,\zeta)) + K(t,x,\zeta)$ for $\zeta \in E$. An approximation to the QVI solution is then obtained for $n \to \infty$.

3.3 Optimal stopping and viscosity solutions of HJBVIs

We prove in this section viscosity solution existence and uniqueness for the HJB variational inequality (HJBVI) associated with combined stochastic control and optimal stopping of jump-diffusions. The general framework and the assumptions are very similar to those of Chapter 2, and much of the introductory discussion in that chapter also applies here.

Viscosity solution existence and uniqueness of (HJB)VIs for diffusions was proved in Øksendal and Reikvam [91], Ceci and Bassan [26]; see also Bensoussan and Lions [16], [17] for solutions in Sobolev spaces. An important reference is certainly Pham [99], who proved viscosity solution existence and uniqueness of HJBVIs for a jump-diffusion up to some terminal time (parabolic problem). Unfortunately, his result does not fit in the framework we have adopted; we consider here the exit time problem under more general conditions than [99]. Furthermore, as Jakobsen and Karlsen [68] argue, in [99] there was a gap in the treatment of viscosity solution uniqueness for a singular integral term.

3.3.1 Setting and HJBVI

The basic probabilistic setting is the same as exposed in §2.2. Again W is an adapted m-dimensional Brownian motion, and $\overline{N}(dz,dt) = N(dz,dt) - 1_{|z|<1}\nu(dz)dt)$ an adapted compensated Poisson random measure. The d-dimensional state process X follows the controlled stochastic differential equation

$$dX_t = \mu(t, X_{t-}, \beta_{t-})\, dt + \sigma(t, X_{t-}, \beta_{t-})\, dW_t + \int_{\mathbb{R}^k} \ell(t, X_{t-}, \beta_{t-}, z)\, \overline{N}(dz, dt) \tag{3.9}$$

for $\mu : \mathbb{R}_0^+ \times \mathbb{R}^d \times B \to \mathbb{R}^d$, $\sigma : \mathbb{R}_0^+ \times \mathbb{R}^d \times B \to \mathbb{R}^{d\times m}$, $\ell : \mathbb{R}_0^+ \times \mathbb{R}^d \times B \times \mathbb{R}^k \to \mathbb{R}^d$ satisfying the necessary conditions such that existence and uniqueness of the SDE is guaranteed. β is a càdlàg adapted stochastic control (where $\beta(t,\omega) \in B$, B compact non-empty metric space), with $\beta \in \mathcal{B} = \mathcal{B}(t,x)$, the admissible set for the stochastic control. Admissible means here in particular that existence and uniqueness of the SDE be guaranteed, and that we only consider Markov controls (see §2.3). We further assume that all constant stochastic controls β in B are admissible, such that $\mathcal{B}(t,x)$ is non-empty.

We define the set of admissible stopping times as $\mathcal{T} = \mathcal{T}(t,x) = \{\tau : \tau \text{ stopping time}, t \leq \tau \leq \tau_S^T\}$. Here τ_S^T is the exit time from some open set $S \subset \mathbb{R}^d$ and some horizon $T \in (t,\infty]$, i.e., $\tau_S^T = \tau_S \wedge T = \inf\{s \geq t : X_s^\beta \notin S\} \wedge T$.

The general combined stochastic control and stopping problem is: Find $\beta \in \mathcal{B}$ and a stopping time $\tau \in \mathcal{T}$ that maximize the payoff starting in t with x

$$J^{(\beta,\tau)}(t,x) = \mathbb{E}^{(t,x)}\left[\int_t^\tau f(s, X_s^\beta, \beta_s)ds + g(\tau, X_\tau^\beta)1_{\tau<\infty}\right], \tag{3.10}$$

where $f : \mathbb{R}_0^+ \times \mathbb{R}^d \times B \to \mathbb{R}$, $g : \mathbb{R}_0^+ \times \mathbb{R}^d \to \mathbb{R}$ are measurable. The value function v of combined stochastic control and stopping is defined by

$$v(t,x) = \sup_{\beta\in\mathcal{B}(t,x),\tau\in\mathcal{T}(t,x)} J^{(\beta,\tau)}(t,x). \tag{3.11}$$

We require the integrability condition on the negative parts of f and g,

$$\mathbb{E}^{(t,x)}\left[\int_t^\tau f^-(s, X_s^\beta, \beta_s)ds + g^-(\tau, X_\tau^\beta)1_{\tau<\infty}\right] < \infty \tag{3.12}$$

for all $\beta \in \mathcal{B}(t,x), \tau \in \mathcal{T}(t,x)$.

Parabolic HJBVI. For a fixed finite horizon $T > 0$, we define $S_T := [0,T) \times S$ and its parabolic nonlocal "boundary" $\partial^+ S_T := \big([0,T) \times (\mathbb{R}^d \setminus S)\big) \cup \big(\{T\} \times \mathbb{R}^d\big)$. Further denote $\partial^* S_T := ([0,T) \times \partial S) \cup \big(\{T\} \times \overline{S}\big)$. The hope is to find the value function by investigating the following parabolic Hamilton-Jacobi-Bellman variational inequality with the so-called *obstacle* g:

$$\begin{aligned} \min(-\sup_{\beta\in B}\{u_t + \mathcal{L}^\beta u + f^\beta\}, u - g) &= 0 \qquad \text{in } S_T \\ u - g &= 0 \qquad \text{in } \partial^+ S_T, \end{aligned} \tag{3.13}$$

for $\mathcal{L}^\beta$ the generator of X in the SDE (3.9) for constant stochastic control β, and $f^\beta(\cdot) := f(\cdot, \beta)$. The generator $\mathcal{L}^\beta$ has the form $(y = (t, x))$

$$\mathcal{L}^\beta u(y) = \frac{1}{2} tr\left(\sigma(y,\beta)\sigma^T(y,\beta) D_x^2 u(y)\right) + \langle \mu(y,\beta), \nabla_x u(y)\rangle + \int u(t, x + \ell(y,\beta,z)) - u(y) - \langle \nabla_x u(y), \ell(y,\beta,z)\rangle 1_{|z|<1}\, \nu(dz). \tag{3.14}$$

The equation for S in (3.13) can be motivated by Dynkin's formula and because $v \geq g$ by immediate stopping; on S^c we have no option to return to S, instead the problem is stopped with payoff g.

We have already established in §2.2 conditions under which the term $\mathcal{L}^\beta u$ in the HJBVI (3.13) is well-defined. Again, we take the Lévy measure ν and ℓ as given, and fix the space $\mathcal{PB}$ of suitably polynomially bounded functions such that for $u \in \mathcal{PB} \cap C^{1,2}([0,T) \times \mathbb{R}^d)$, the term $\sup_{\beta \in B}\{u_t + \mathcal{L}^\beta u + f^\beta\}$ is well-defined. The remarks on $\mathcal{PB}$ in §2.2 apply in the same way. The space $\mathcal{PB}_p = \mathcal{PB}_p([0,T] \times \mathbb{R}^d)$ consists of all functions $u \in \mathcal{PB}$, for which there is a constant C such that $|u(t,x)| \leq C(1 + |x|^p)$ for all $(t,x) \in [0,T] \times \mathbb{R}^d$.

Elliptic HJBVI. For finite time horizon T, (3.13) is investigated on $[0,T] \times \mathbb{R}^d$ (parabolic problem). For $T = \infty$, typically a discounting factor $e^{-\rho(t+s)}$ for $\rho > 0$ applied to the functions f and g takes care of the well-definedness of the value function, e.g., $f(t,x,\beta) = e^{-\rho(t+s)} \tilde{f}(x,\beta)$; furthermore, the SDE coefficients in (3.9) must be time-independent. In this time-independent case, a transformation of the type $u(t,x) = e^{-\rho(t+s)}\tilde{u}(x)$ gives us the elliptic HJBVI to investigate

$$\begin{aligned} \min(-\sup_{\beta \in B}\{-\rho u + \mathcal{L}^\beta u + f^\beta\}, u - g) &= 0 \qquad \text{in } S \\ u - g &= 0 \qquad \text{in } \mathbb{R}^d \setminus S, \end{aligned} \tag{3.15}$$

where the functions and variables have been appropriately renamed, and

$$\mathcal{L}^\beta u(x) = \frac{1}{2} tr\left(\sigma(x,\beta)\sigma^T(x,\beta) D^2 u(x)\right) + \langle \mu(x,\beta), \nabla u(x)\rangle + \int u(x + \ell(x,\beta,z)) - u(x) - \langle \nabla u(x), \ell(x,\beta,z)\rangle 1_{|z|<1}\, \nu(dz). \tag{3.16}$$

Dynamic programming inequality. As in §2.3, one can derive a dynamic programming inequality based on the strong Markov property of the controlled process until the stopping time $t \leq \tau \leq \tau_S^T$. For any stopping time $\tilde{\tau}$ with $t \leq \tilde{\tau} \leq \tau$,

$$\begin{aligned} J^{(\beta,\tau)}(t,x) &= \mathbb{E}^{(t,x)}\left\{\int_t^{\tilde{\tau}} f(s, X_s^\beta, \beta_s) ds + \mathbb{E}^{(\tilde{\tau}, X_{\tilde{\tau}}^\beta)}\left[\int_{\tilde{\tau}}^{\tau} f(s, X_s^\beta, \beta_s) ds + g(\tau, X_\tau^\beta) 1_{\tau<\infty}\right]\right\} \\ &= \mathbb{E}^{(t,x)}\left\{\int_t^{\tilde{\tau}} f(s, X_s^\beta, \beta_s) ds + J^{(\beta,\tau)}(\tilde{\tau}, X_{\tilde{\tau}}^\beta)\right\} \qquad (3.17) \\ &\leq \mathbb{E}^{(t,x)}\left\{\int_t^{\tilde{\tau}} f(s, X_s^\beta, \beta_s) ds + v(\tilde{\tau}, X_{\tilde{\tau}}^\beta)\right\}. \qquad (3.18) \end{aligned}$$

As in the impulse control case, for our proofs of viscosity existence and uniqueness we do not need to formulate the dynamic programming equation.

3.3.2 (Strong) viscosity solution existence and uniqueness

In this subsection, we will prove the viscosity existence result for the parabolic HJBVI (3.13), whereas the viscosity uniqueness is proved for the elliptic case (3.15). This is convenient for the statement of the proofs; the corresponding results for the other case can be deduced as in §§ 2.4 and 2.5.

The assumptions needed to prove viscosity existence and uniqueness of (3.13) are mostly the same as in §2.2 for the HJBQVI, with the following differences: (V1) and (V2) can be removed because they only refer to impulses; in (E3), the first (lim inf) condition can be dropped, and the second (lim sup) needs to hold without the impulse condition. As well, the discussion of the assumptions still holds, except that it is easier to find a strict supersolution w for optimal stopping.

Recall the definition of $S_T := [0,T)\times S$ and its parabolic "boundary" $\partial^+ S_T := \big([0,T) \times (\mathbb{R}^d \setminus S)\big) \cup \big(\{T\} \times \mathbb{R}^d\big)$. Let us also recall the definition of $\mathcal{PB}$ encapsulating the growth condition from §2.2.

Assumption 3.3.1. *Let (V3) and (V4) from Assumption 2.2.1 be satisfied.*

Assumption 3.3.2. *Let (E1), (E2) and (E4) from Assumption 2.2.2 hold. Furthermore, the value function v satisfies for every $(t,x) \in \partial^* S_T$, and all sequences $(t_n,x_n)_n \subset [0,T) \times S$ converging to (t,x),*

$$\limsup_{n\to\infty} v(t_n,x_n) \le g(t,x).$$

With the above modified assumptions, the following theorem holds (the precise definition of viscosity solution is introduced below):

Theorem 3.3.1 (Viscosity existence and uniqueness for parabolic HJBVI)**.** *Let Assumptions 3.3.1, 3.3.2, 2.2.3 and 2.2.4 be satisfied. Assume further that $v \in \mathcal{PB}_p([0,T] \times \mathbb{R}^d)$, and that there is a nonnegative $w \in \mathcal{PB} \cap C^{1,2}([0,T] \times \mathbb{R}^d)$ with $w(t,x)/|x|^p \to \infty$ for $|x| \to \infty$ (uniformly in t) and a constant $\kappa > 0$ such that*

$$\begin{aligned} \min(-\sup_{\beta\in B}\{w_t + \mathcal{L}^\beta w + f^\beta\}, w - g) &\ge \kappa \qquad \text{in } S_T \\ w - g &\ge \kappa \qquad \text{in } \partial^+ S_T. \end{aligned} \tag{3.19}$$

Then the value function v in (3.11) *is the unique viscosity solution in $\mathcal{PB}_p([0,T] \times \mathbb{R}^d)$ of the parabolic HJBVI (3.13), and it is continuous on $[0,T] \times \mathbb{R}^d$.*

For the proof of Theorem 3.3.1, see Theorem 3.3.3 and (for uniqueness in the elliptic case) Theorem 3.3.4.

Under the time-independent version of Assumptions 2.2.1 - 2.2.4, an essentially identical existence and uniqueness result holds for the elliptic HJBVI (3.15). We refrain from repeating it, but instead refer to Theorem 3.3.4, where the uniqueness is discussed in the elliptic case.

Existence

We consider in this section the parabolic HJBVI (3.13) with the integro-differential operator $\mathcal{L}^\beta$ from (3.14) (or infinitesimal generator of the process X).

Let us now define what exactly we mean by a viscosity solution of (3.13). Let $LSC(\Omega)$ (resp., $USC(\Omega)$) denote the set of measurable functions on the set Ω that are lower semicontinuous (resp., upper semicontinuous). Let $T > 0$, and let u^* (u_*) define the upper (lower) semicontinuous envelope of a function u on $[0,T] \times \mathbb{R}^d$, i.e. the limit superior (limit inferior) is taken only from within this set.

Definition 3.3.2 (Viscosity solution of HJBVI). *A function $u \in \mathcal{PB}([0,T] \times \mathbb{R}^d)$ is a (viscosity) subsolution of (3.13) if for all $(t_0, x_0) \in [0,T] \times \mathbb{R}^d$ and $\varphi \in \mathcal{PB} \cap C^{1,2}([0,T) \times \mathbb{R}^d)$ with $\varphi(t_0,x_0) = u^*(t_0,x_0)$, $\varphi \geq u^*$ on $[0,T) \times \mathbb{R}^d$,*

$$\begin{aligned} \min\left(-\sup_{\beta\in B}\left\{\frac{\partial\varphi}{\partial t} + \mathcal{L}^\beta\varphi + f^\beta\right\}, u^* - g\right) &\leq 0 \qquad \text{in } (t_0,x_0) \in S_T \\ u^* - g &\leq 0 \qquad \text{in } (t_0,x_0) \in \partial^+ S_T. \end{aligned}$$

A function $u \in \mathcal{PB}([0,T] \times \mathbb{R}^d)$ is a (viscosity) supersolution of (3.13) if for all $(t_0,x_0) \in [0,T] \times \mathbb{R}^d$ and $\varphi \in \mathcal{PB} \cap C^{1,2}([0,T) \times \mathbb{R}^d)$ with $\varphi(t_0,x_0) = u_*(t_0,x_0)$, $\varphi \leq u_*$ on $[0,T) \times \mathbb{R}^d$,

$$\begin{aligned} \min\left(-\sup_{\beta\in B}\left\{\frac{\partial\varphi}{\partial t} + \mathcal{L}^\beta\varphi + f^\beta\right\}, u_* - g\right) &\geq 0 \qquad \text{in } (t_0,x_0) \in S_T \\ u_* - g &\geq 0 \qquad \text{in } (t_0,x_0) \in \partial^+ S_T. \end{aligned}$$

A function u is a viscosity solution if it is sub and supersolution.

Note immediately from the definition that any viscosity solution of (3.13) satisfies $u = g$ on $\partial^+ S_T$. Now we can state the existence theorem:

Theorem 3.3.3 (HJBVI viscosity solution: existence). *Let Assumptions 3.3.1 and 3.3.2 be satisfied. Then the value function v in* (3.11) *is a viscosity solution of the parabolic HJBVI (3.13) as defined above.*

Proof: Similar to that of Theorem 2.4.2. We start with the supersolution proof. By the necessary condition $v \geq g$ on $[0,T] \times \mathbb{R}^d$ and continuity of g (E2), we have $v_* \geq g$ on $[0,T] \times \mathbb{R}^d$. In a fixed $(t_0,x_0) \in [0,T) \times S$, for $\varphi \in \mathcal{PB} \cap C^{1,2}([0,T) \times \mathbb{R}^d)$, $\varphi(t_0,x_0) = v_*(t_0,x_0)$, $\varphi \leq v_*$ on $[0,T) \times \mathbb{R}^d$, the inequality

$$-\sup_{\beta\in B}\left\{\frac{\partial\varphi}{\partial t} + \mathcal{L}^\beta\varphi + f^\beta\right\} \geq 0$$

is proved as in Theorem 2.4.2 by taking as control a constant β, and $\tau = \tau_S^T$ ("no impulses"). We note that (E3) was not needed so far.

Now consider the subsolution case. First, the inequality $v^* \leq g$ holds by the version of (E3) on $\partial^* S_T$ and by the necessary condition $v \leq g$ also on the rest of $\partial^+ S_T$.

Let $(t_0,x_0) \in S_T$ and $\varphi \in \mathcal{PB} \cap C^{1,2}([0,T) \times \mathbb{R}^d)$ such that $v^*(t_0,x_0) = \varphi(t_0,x_0)$ and $\varphi \geq v^*$ on $[0,T) \times \mathbb{R}^d$. If $v^*(t_0,x_0) \leq g(t_0,x_0)$, then the subsolution inequality holds trivially. So consider from now on the case $v^*(t_0,x_0) > g(t_0,x_0)$.

To argue by contradiction, we assume that there is an $\eta > 0$ such that in $(t_0,x_0) \in [0,T) \times S$,

$$\sup_{\beta\in B}\left\{\frac{\partial\varphi}{\partial t} + \mathcal{L}^\beta\varphi + f^\beta\right\} < -\eta < 0. \tag{3.20}$$

From the definition of v^*, there exists a sequence $(t_n, x_n) \subset S_T$ such that $(t_n, x_n) \to (t_0, x_0)$, $v(t_n, x_n) \to v^*(t_0, x_0)$ for $n \to \infty$. By continuity of φ, the term $\delta_n := v(t_n, x_n) - \varphi(t_n, x_n)$ converges to 0 as n goes to infinity.

We choose for each $n \in \mathbb{N}$ an admissible ε_n-optimal control pair $\beta^n \in \mathcal{B}$, $t_n \leq \tau^n \leq \tau_S^T$, where $\varepsilon_n \downarrow 0$. Define for some small $\rho > 0$ the stopping time $\bar{\tau}_n := \bar{\tau}_n^\rho \wedge \tau^n$, where

$$\bar{\tau}_n^\rho := \inf\{s \geq t_n : |X_s^{\beta^n, t_n, x_n} - x_n| \geq \rho\} \wedge (t_n + \rho).$$

By applying Dynkin's formula to the dynamic programming inequality (3.18) (with v replaced by φ) until $\bar{\tau}_n$, we conclude as in Th. 2.4.2 that $\bar{\tau}_n \to t_0$ in probability. On the other hand, by combining (3.18) with the ε_n-optimality, we have

$$v(t_n, x_n) \leq \mathbb{E}^{(t,x)} \left[\int_t^{\bar{\tau}_n} f(s, X_s^\beta, \beta_s) ds + v(\bar{\tau}_n, X_{\bar{\tau}_n}^\beta) \right] + \varepsilon_n.$$

As in the proof of Th. 2.4.2, we split Ω into disjoint sets: (I) $\{\bar{\tau}_n^\rho < \tau^n\}$, i.e., exit before stopping; (II) $\{\bar{\tau}_n^\rho \geq \tau^n\} \cap A_n(\rho)^c$, i.e., exit not before stopping, process unbounded; (III) $\{\bar{\tau}_n^\rho \geq \tau^n\} \cap A_n(\rho)$, i.e., exit not before stopping, process contained in $B(x_n, \rho)$. We know from Th. 2.4.2 that $1_{(III)} \to 1$ a.s. for $n \to \infty$. In cases (II) and (III), we stop before leaving $A_n(\rho)$ and get g as payoff, so

$$\begin{aligned} v(t_n, x_n) \quad &\leq \sup_{\substack{|t'-t_0|<\rho \\ |y'-x_0|<\rho}} f(t', y', \beta_{t'}) \mathbb{E}[\bar{\tau}_n - t_n] + \mathbb{E}[|v(\bar{\tau}_n, X_{\bar{\tau}_n}^\beta)| 1_{(I)}] \\ &\quad + \mathbb{E}[|g(\bar{\tau}_n, X_{\bar{\tau}_n}^\beta)| 1_{(II)}] + \sup_{\substack{|t'-t_0|<\rho \\ |y'-x_0|<\rho}} g(t', y') \, \mathbb{E} 1_{(III)} + \varepsilon_n. \end{aligned}$$

By dominated convergence we obtain as in Th. 2.4.2 for $n \to \infty$ and then $\rho \to 0$

$$v^*(t_0, x_0) \leq \lim_{\rho \downarrow 0} \sup_{\substack{|t'-t_0|<\rho \\ |y'-x_0|<\rho}} g(t', y') = g(t_0, x_0),$$

a contradiction. $\square$

Uniqueness

We state the comparison and uniqueness results for the elliptic case similar to §2.5. The definition of viscosity solution for the elliptic case is a straightforward adaptation of the parabolic case. All results in §2.5 pertaining only to the HJB part $-\sup_{\beta \in B}\{\mathcal{L}^\beta u + f^\beta\}$ (or F) are still true, where $\mathcal{L}^\beta$ is the elliptic operator defined in (3.16). The equivalence of different viscosity solution definitions concerned only the HJB part.

A maximum principle analogous to Corollary 2.5.9 holds because of the continuity of g (E2) in S, as is easy to see. Lemma 2.5.10 also applies for our case under the assumption that there is a $w \in \mathcal{PB} \cap C^2(\mathbb{R}^d)$ and a positive function $\kappa : \mathbb{R}^d \to \mathbb{R}$ such that

$$\begin{aligned} \min(-\sup_{\beta \in B}\{-\rho w + \mathcal{L}^\beta w + f^\beta\}, w - g) \geq \kappa \qquad &\text{in } S \\ w - g \geq \kappa \qquad &\text{in } \mathbb{R}^d \setminus S, \end{aligned} \tag{3.21}$$

simply because the operator $Gu :\equiv g$ satisfies (anti-)convexity conditions just as $\mathcal{M}$ in the impulse control case. Now we can state the comparison result:

Theorem 3.3.4. *Let Assumptions 3.3.1, 2.2.3 and 2.2.4 be satisfied. Assume further that there is a* $w \geq 0$ *as in* (3.21) *(for a constant* $\kappa > 0$*) with* $w(x)/|x|^p \to \infty$ *for* $|x| \to \infty$*. If* $u \in \mathcal{PB}_p(\mathbb{R}^d)$ *is a subsolution and* $v \in \mathcal{PB}_p(\mathbb{R}^d)$ *a supersolution of (3.15), then* $u^* \leq v_*$*.*

Corollary 3.3.5 (HJBVI viscosity solution: uniqueness)**.** *Under the same assumptions, there is at most one viscosity solution in* $\mathcal{PB}_p(\mathbb{R}^d)$ *of (3.15), and it is continuous.*

Proof of Theorem 3.3.4: Essentially the same proof as for Th. 2.5.11. Let us therefore only shortly sketch the differences. We prove by contradiction, and assume $M := \sup_{x\in\mathbb{R}^d} u_m(x) - v_m(x) > 0$. Step 1 reduces to the part involving g. Step 2 begins in the same way as in the proof of Th. 2.5.11, in Case 2a ($u_m(x_\varepsilon) - g(x_\varepsilon) \leq 0$), we have by $v_m(y_\varepsilon) - g(y_\varepsilon) \geq \frac{\kappa}{m}$

$$M = \limsup_{\varepsilon\to 0}(u_m(x_\varepsilon) - v_m(y_\varepsilon)) \leq \limsup_{\varepsilon\to 0} g(x_\varepsilon) - \liminf_{\varepsilon\to 0} g(y_\varepsilon) - \frac{\kappa}{m} = -\frac{\kappa}{m},$$

which is a contradiction. From Case 2b on, the proof is again the same as for Th. 2.5.11. □

3.3.3 Weak viscosity solutions

Let us now state and prove a weaker version of the viscosity solution result for optimal stopping, which does not directly require continuity on the nonlocal boundary $\partial^+ S_T$. For this we need a new definition of viscosity solution, which we call "weak viscosity solution" as opposed to Definition 3.3.2, which is sometimes called "strong viscosity solution". The concept of this weak viscosity solution makes sense for optimal stopping because there is no nonlocal feedback term in $\partial^+ S_T$ as in the impulse control case.

Definition 3.3.6 (Weak viscosity solution of HJBVI)**.** *A function* $u \in \mathcal{PB}([0,T]\times\mathbb{R}^d)$ *is a weak (viscosity) subsolution of (3.13) if for all* $(t_0,x_0) \in [0,T]\times\mathbb{R}^d$ *and* $\varphi \in \mathcal{PB}\cap C^{1,2}([0,T)\times\mathbb{R}^d)$ *with* $\varphi(t_0,x_0) = u^*(t_0,x_0)$*,* $\varphi \geq u^*$ *on* $[0,T)\times\mathbb{R}^d$*,*

$$\min\left(-\sup_{\beta\in B}\left\{\frac{\partial\varphi}{\partial t} + \mathcal{L}^\beta\varphi + f^\beta\right\}, u^* - g\right) \quad \leq \quad 0 \qquad \textit{in } (t_0,x_0) \in S_T,$$

and $u - g \leq 0$ *in* $\partial^+ S_T$*.*
A function $u \in \mathcal{PB}([0,T]\times\mathbb{R}^d)$ *is a weak (viscosity) supersolution of (3.13) if for all* $(t_0,x_0) \in [0,T]\times\mathbb{R}^d$ *and* $\varphi \in \mathcal{PB}\cap C^{1,2}([0,T)\times\mathbb{R}^d)$ *with* $\varphi(t_0,x_0) = u_*(t_0,x_0)$*,* $\varphi \leq u_*$ *on* $[0,T)\times\mathbb{R}^d$*,*

$$\min\left(-\sup_{\beta\in B}\left\{\frac{\partial\varphi}{\partial t} + \mathcal{L}^\beta\varphi + f^\beta\right\}, u_* - g\right) \quad \geq \quad 0 \qquad \textit{in } (t_0,x_0) \in S_T,$$

and $u - g \geq 0$ *in* $\partial^+ S_T$*.*
A function u *is a weak viscosity solution if it is a weak sub and supersolution.*

Obviously, a weak viscosity solution of (3.13) which is continuous on $[0,T]\times\mathbb{R}^d$ is also a strong viscosity solution, and any strong viscosity solution is a weak one. In return, we are only able to prove existence of a viscosity solution.

Compared to the strong viscosity results in §3.3.2, we can drop the assumption of continuity of v at $\partial^* S_T$, and only need continuity of g in S_T:

Assumption 3.3.3. *Let (E1) and (E4) from Assumption 2.2.2 hold. Furthermore, the function g is continuous on S_T.*

With these modified assumptions, the following theorem holds (again there is a corresponding elliptic version with the usual adjustments):

Theorem 3.3.7 (Weak viscosity existence for parabolic HJBVI)**.** *Let Assumptions 3.3.1 and 3.3.3 be satisfied. Then the value function v in* (3.11) *is a weak viscosity solution of the parabolic HJBVI (3.13).*

Proof: The existence proof of Th. 3.3.3 changes only in that it is unnecessary to consider the upper or lower semicontinuous envelope in $\partial^+ S_T$, and thus the continuity requirements for v and g in $\partial^+ S_T$ can be dropped. □

The proof of the uniqueness result fails because of problems with the lower and upper semicontinuous envelopes. However, from the comparison for strong viscosity solutions, it follows that there is at most one weak viscosity solution in $\mathcal{PB}_p \cap C([0,T] \times \mathbb{R}^d)$.

3.3.4 A stability result

Stability results are important tools not only for viscosity solution theory. *Stability* in general means that a solution of an equation is stable with respect to small variations of functions or parameters. We discussed the typical form a stability result takes in viscosity solution theory already in §1.3.

We state in this subsection a stability result for the HJBVI of combined stochastic control and stopping, if the obstacle g converges uniformly on compacts in S. For a similar result for integro-differential HJB equations see Barles and Imbert [11], or the somewhat more restrictive Lemma II.6.3 in Fleming and Soner [46]. We will need the stability result in §3.4.2 to prove the convergence of iterated optimal stopping.

Consider the disturbed elliptic HJBVI with (possibly discontinuous) obstacle g^ε:

$$\begin{aligned} \min(-\sup_{\beta\in B}\{-\rho u + \mathcal{L}^\beta u + f^\beta\}, u - g^\varepsilon) &= 0 \qquad \text{in } S \\ u - g^\varepsilon &= 0 \qquad \text{in } \mathbb{R}^d \setminus S, \end{aligned} \tag{3.22}$$

where $\mathcal{L}^\beta$ is given by (3.16). We formulate theorem and proof for the elliptic setting to alleviate notation; essentially the same result holds true for the parabolic HJBVI with virtually unchanged proof.

Theorem 3.3.8. *Let Assumptions 3.3.1 and 2.2.3 be satisfied. For $\varepsilon > 0$, let u^ε be a continuous (strong) viscosity solution of* (3.22)*, and assume that $(u^\varepsilon)_\varepsilon \subset \mathcal{PB}(\mathbb{R}^d)$ uniformly, i.e., there is a $C > 0$ such that*

$$|u^\varepsilon(x)| \le C(1 + R(x)) \qquad \forall \varepsilon > 0.$$

If $u^\varepsilon \to u$ uniformly on compacts, and $g^\varepsilon \to g$ uniformly on compacts in S and pointwise in $\mathbb{R}^d \setminus S$ for $\varepsilon \to 0$, then u is a continuous (strong) viscosity solution of (3.22) *with obstacle g.*

For the following proof, we will use the first of the equivalent viscosity solution definitions in §2.5.2. We recall the notation

$$F(x_0, u, \nabla\varphi, D^2\varphi, \mathcal{I}_\beta[x_0, \varphi(\cdot)]) = -\sup_{\beta\in B}\{-\rho\varphi(x_0) + \mathcal{L}^\beta\varphi(x_0) + f^\beta\}.$$

Proof of Th. 3.3.8: First note that on $\mathbb{R}^d \setminus S$, if $u^\varepsilon = g^\varepsilon$ for all $\varepsilon > 0$, then also $u = g$. Thus it remains to consider points in S. As uniform limit of continuous functions, u is continuous, too.

We first prove that u is subsolution. Take $x_0 \in S$, and $\varphi \in \mathcal{PB} \cap C^2(\mathbb{R}^d)$ such that $u(x_0) = \varphi(x_0)$ and $\varphi \geq u$. Wlog, we may even assume $\varphi > u$ in $\mathbb{R}^d \setminus \{x_0\}$ (see §2.5). We have to show that in x_0,

$$\min\big(F(x_0, u, \nabla\varphi, D^2\varphi, \mathcal{I}_\beta[x_0, \varphi(\cdot)]), u - g\big) \leq 0$$

If $u(x_0) - g(x_0) \leq 0$, then nothing is to prove. If $u(x_0) - g(x_0) > 0$, then by the locally uniform convergence, there is a neighbourhood $U \ni x_0$ where $u^\varepsilon - g^\varepsilon > 0$ for small enough $\varepsilon > 0$.

Because of locally uniform convergence, for any compact K with $x_0 \in int(K)$, there is a sequence $x_\varepsilon \to x_0$ for $\varepsilon \to 0$ such that $u^\varepsilon - \varphi$ assumes in x_ε its maximum on K. Setting $\varphi^\varepsilon := \varphi - \varphi(x_\varepsilon) + u^\varepsilon(x_\varepsilon)$, we have $u^\varepsilon(x_\varepsilon) = \varphi^\varepsilon(x_\varepsilon)$ and $\varphi^\varepsilon \geq u^\varepsilon$ on K. Choose $K = K_{r+1} = \overline{B(x_0, r+1)}$ for $r > 0$ in the above. We can extend φ^ε from K_r to a function $\tilde{\varphi}^\varepsilon$, with $\tilde{\varphi}^\varepsilon \geq u^\varepsilon$ globally, and $(\tilde{\varphi}^\varepsilon) \subset \mathcal{PB} \cap C^2(\mathbb{R}^d)$ uniformly. Let $\delta > 0$, and choose r large enough such that

$$\int_{\ell(x_\varepsilon,\beta,z)\notin K_r} (1 + R(x_\varepsilon + \ell(x_\varepsilon, \beta, z)))\ \nu(dz) \leq \delta \tag{3.23}$$

uniformly in ε small (possible by (V3), (V4)). Then there is a constant C (independent of small ε, δ) such that $|\mathcal{I}_\beta[x_\varepsilon, \varphi^\varepsilon(\cdot)] - \mathcal{I}_\beta[x_\varepsilon, \tilde{\varphi}^\varepsilon(\cdot)]| \leq C\delta$ for all $\beta \in B$. In $x_\varepsilon \in U \cap S$, we can apply the viscosity subsolution property for $\tilde{\varphi}^\varepsilon$ and obtain using the estimate (3.23)

$$F(x_\varepsilon, u, \nabla\varphi^\varepsilon, D^2\varphi^\varepsilon, \mathcal{I}_\beta[x_\varepsilon, \varphi^\varepsilon(\cdot)]) \leq C\delta,$$

which by Prop. 2.5.1 converges in a subsequence to $F(x_0, u, \nabla\varphi, D^2\varphi, \mathcal{I}_\beta[x_0, \varphi(\cdot)]) \leq C\delta$ for $\varepsilon \to 0$. (The argument is carried out as in Cor. 2.5.9 by choosing a convergent subsequence of the maximizer β_ε in F.) Finally, let δ converge to 0.

The supersolution proof proceeds along the same lines. For $x_0 \in S$, the inequality $u - g \geq 0$ holds as limit of $u^\varepsilon - g^\varepsilon$, the rest is similar to the above. □

3.4 Iterated optimal stopping

After showing the equivalence of different reformulations of the optimal stopping iteration, we will prove convergence in a viscosity solution sense (uniformly on compacts).

3.4.1 Definition and equivalent reformulation

Iterated optimal stopping is intuitive to understand: Start with the value function v^0 of the problem without impulses. In the first iteration, stop optimally to get the payoff $\mathcal{M}v^0$ (or do not stop at all) and call the resulting value function v^1. Iterate this procedure to obtain the iterates $(v^n)_{n\geq 1}$, where v^n contains n optimally placed impulses. One would expect the sequence $(v^n)_{n\geq 1}$ to be monotonically increasing.

The subtle point about this procedure is that it may not be optimal to do an impulse at all (see Example 3.4.1). We make sure that the controller is not forced to act by defining (in the time-dependent case for $T > t$)

$$\begin{aligned}
v^0(t,x) &:= \sup_{\beta\in\mathcal{B}} \mathbb{E}^{(t,x)}\left[\int_t^{\tau_S^T} f(s,X_s^\beta,\beta_s)ds + g(\tau_S^T, X^\beta_{\tau_S^T})\right] \\
v^n(t,x) &:= Qv^{n-1}(t,x) \\
&:= \sup_{\beta\in\mathcal{B},\tau\in\mathcal{T}} \mathbb{E}^{(t,x)}\left[\int_t^{\tau} f(s,X_s^\beta,\beta_s)ds + \mathcal{M}v^{n-1}(\tau,X_\tau^\beta)1_{\tau<\tau_S^T}\right. \\
&\qquad\qquad \left. + \max\left(g,\mathcal{M}v^{n-1}\right)(\tau,X_\tau^\beta)1_{\tau=\tau_S^T}\right], \quad n\geq 1,
\end{aligned} \tag{3.24}$$

where $\mathcal{M}$ denotes the impulse intervention operator,

$$\mathcal{M}u(t,x) = \sup\{u(t,\Gamma(t,x,\zeta)) + K(t,x,\zeta) : \zeta\in Z(t,x)\}.$$

If the optimal stopping event the controller defined does not occur, then he will simply get the payoff g without an impulse. To ensure well-definedness, we assume that

$$\mathbb{E}^{(t,x)}\left[\int_t^\tau f^-(s,X_s^\beta,\beta_s)ds + (\mathcal{M}v^0)^-(\tau,X_\tau^\beta)1_{\tau<\tau_S^T} + \max\left(g^-,(\mathcal{M}v^0)^-\right)(\tau,X_\tau^\beta)1_{\tau=\tau_S^T}\right] < \infty$$

for all $\beta\in\mathcal{B}$, $\tau\in\mathcal{T}$. The start iterate v^0 is well-defined by assuming (3.12).

The definition in (3.24) above can be interpreted also in the infinite-horizon case with $T=\infty$, if it is implicitly understood that no terminal payoff takes place if $\tau(\omega)=\infty$.

It is not immediate from the definition that the sequence $(v^n)_{n\geq 1}$ is increasing. The following proposition shows that this is actually the case:

Proposition 3.4.1. *The sequence $(v^n)_{n\geq1}$ defined in (3.24) is monotonically increasing, i.e.,*

$$v^{n-1}\leq v^n \qquad \forall n\in\mathbb{N}.$$

Furthermore, the operator Q mapping v^{n-1} into v^n is monotone.

Proof: We use here extensively the monotonicity property of $\mathcal{M}$, proved in Lemma 2.4.3. First, by setting $\tau=\tau_S^T$, we have

$$\begin{aligned}
v^1(t,x) &= \sup_{\beta\in\mathcal{B},\tau\in\mathcal{T}} \mathbb{E}^{(t,x)}\left[\int_t^\tau f(s,X_s^\beta,\beta_s)ds + \mathcal{M}v^0(\tau,X_\tau^\beta)1_{\tau<\tau_S^T}\right. \\
&\qquad\qquad \left. + \max\left(g,\mathcal{M}v^0\right)(\tau,X_\tau^\beta)1_{\tau=\tau_S^T}\right] \\
&\geq \sup_{\beta\in\mathcal{B}} \mathbb{E}^{(t,x)}\left[\int_t^\tau f(s,X_s^\beta,\beta_s)ds + g(\tau_S^T,X^\beta_{\tau_S^T})\right] \\
&= v^0(t,x)
\end{aligned}$$

by definition of v^0. The monotonicity of Q follows immediately from the corresponding property of $\mathcal{M}$, and thus for $n\geq 2$,

$$v^n = Q^{n-1}v^1 \geq Q^{n-1}v^0 = v^{n-1}.$$

□

From the proof of Prop. 3.4.1, we see that the increasingness property holds mainly because $v^1 \geq v^0$ holds (and the iteration for $n \geq 2$ is defined in the same way as for $n = 1$). So similar iterations can still yield an increasing sequence, provided it is increasing for the first impulse. We will see later in this subsection that we can alternatively write as payoff $\max(v^{n-1}, \mathcal{M}v^{n-1})$ inside the expectation of (3.24).

For the further investigation, we need the viscosity properties of v^n. The HJBVI associated with (3.24) for $n \geq 1$ is

$$\begin{aligned} \min(-\sup_{\beta \in B}\{u_t + \mathcal{L}^\beta u + f^\beta\}, u - \mathcal{M}v^{n-1}) = 0 \qquad & \text{in } S_T \\ u - \max(g, \mathcal{M}v^{n-1}) = 0 \qquad & \text{in } \partial^+ S_T, \end{aligned} \tag{3.25}$$

for the SDE generator $\mathcal{L}^\beta$ defined in (3.14). At this point, we are only able to prove weak viscosity solution existence (see §3.3.3) due to the discontinuity of the obstacle.

The result and the needed assumptions follow immediately from the general result of Theorem 3.3.7. We state the assumptions explicitly:

Proposition 3.4.2. *Let (V3), (V4), (E4) be satisfied, and assume that $v^n \in \mathcal{PB}([0,T] \times \mathbb{R}^d)$ and that $\mathcal{M}v^{n-1}$ continuous in S_T. Then the optimal stopping iterate v^n defined in* (3.24) *for $n \geq 1$ is a weak viscosity solution of* (3.25).

Example 3.4.1. In the following trivial deterministic example, we can see that two impulses can be suboptimal: The one-dimensional process X follows $dX_t = 0$, and we want to minimize the payoff $\mathbb{E}^{(t,x)}[(X_T^\gamma)^2]$ at some future time $T > t$ by choosing an optimal impulse control strategy γ, which can jump to any point via $\Gamma(x, \zeta) = x - \zeta$ with fixed costs $k > 0$. It is easy to see that it is optimal to jump at most once for each path, and thus the value function v is equal to

$$v(t,x) = \begin{cases} k & x^2 \geq k \\ x^2 & x^2 < k \end{cases}.$$

The iteration defined in (3.24) converges already in one step, i.e. $v^1 = v$; for starting values $|x| < \sqrt{k}$, the optimal strategy in the first iteration is to choose $\tau = T$. If, however, the optimal stopping iteration were defined by

$$\hat{v}^n(t,x) = \inf_{\tau \leq \tau_S^T} \mathbb{E}^{(t,x)} \left[\int_t^\tau f(s, X_s)ds + \mathcal{M}\hat{v}^{n-1}(\tau, X_\tau) \right], \quad n \geq 1,$$

then $\hat{v}^n(t,x) = n \cdot k$ for $n \geq 1$, which is not a decreasing sequence as it should be. If we allowed to skip the impulse in the above formula (with no payoff instead), we would end up with a discontinuous $\hat{v}^n$.

Equivalent reformulation

We define now a seemingly different optimal stopping iteration by

$$\begin{aligned} \tilde{v}^0(t,x) &:= \sup_{\beta \in \mathcal{B}} \mathbb{E}^{(t,x)} \left[\int_t^{\tau_S^T} f(s, X_s^\beta, \beta_s)ds + g(\tau_S^T, X_{\tau_S^T}^\beta) \right] \\ \tilde{v}^n(t,x) &:= \sup_{\beta \in \mathcal{B}, \tau \in \mathcal{T}} \mathbb{E}^{(t,x)} \left[\int_t^\tau f(s, X_s^\beta, \beta_s)ds + \max\left(\tilde{v}^{n-1}, \mathcal{M}\tilde{v}^{n-1}\right)(\tau, X_\tau^\beta) \right], \quad n \geq 1. \end{aligned} \tag{3.26}$$

Our objective in the remainder of this subsection is to prove that the iterations (3.24) and (3.26) are equivalent, i.e., that $\tilde{v}^n = v^n$ for all $n \in \mathbb{N}_0$. Both iterations have their advantages: It turns out that the numerical computation of $\tilde{v}^n$ is faster (see §3.5), whereas the definition of v^n is more amenable to the convergence proof in §3.4.2; in this proof, we will use the continuity of v^n that follows from the equivalence as a byproduct.

First of all, note that $\tilde{v}^n$ for $n \geq 1$ is characterized as viscosity solution of the HJBVI

$$\begin{aligned} \min(-\sup_{\beta\in B}\{u_t + \mathcal{L}^\beta u + f^\beta\}, u - \max(\tilde{v}^{n-1}, \mathcal{M}\tilde{v}^{n-1})) = 0 \qquad & \text{in } S_T \\ u - \max(\tilde{v}^{n-1}, \mathcal{M}\tilde{v}^{n-1}) = 0 \qquad & \text{in } \partial^+ S_T, \end{aligned} \tag{3.27}$$

for the SDE generator $\mathcal{L}^\beta$ defined in (3.14). The result and the needed assumptions follow immediately from the general result of Theorem 3.3.1. We state the assumptions explicitly:

Proposition 3.4.3. *Let Assumptions 2.2.3 and 2.2.4, as well as (V3), (V4), (E4) be satisfied, and assume that $\tilde{v}^n \in \mathcal{PB}_p([0,T]\times\mathbb{R}^d)$ and that $\tilde{v}^{n-1}$, $\mathcal{M}\tilde{v}^{n-1}$ are continuous. Further assume for any point $(t_0,x_0) \in \partial^* S_T$ and sequence $S_T \ni (t_k,x_k) \to (t_0,x_0)$ that $\limsup_{k\to\infty} \tilde{v}^n(t_k,x_k) \leq \max(\tilde{v}^{n-1}, \mathcal{M}\tilde{v}^{n-1})(t_0,x_0)$, and that there is a nonnegative $w \in \mathcal{PB} \cap C^{1,2}([0,T]\times\mathbb{R}^d)$ with $w(t,x)/|x|^p \to \infty$ for $|x| \to \infty$ (uniformly in t) and a constant $\kappa > 0$ such that*

$$\begin{aligned} \min(-\sup_{\beta\in B}\{w_t + \mathcal{L}^\beta w + f^\beta\}, w - \max(\tilde{v}^{n-1}, \mathcal{M}\tilde{v}^{n-1})) \geq \kappa \qquad & \text{in } S_T \\ w - \max(\tilde{v}^{n-1}, \mathcal{M}\tilde{v}^{n-1}) \geq \kappa \qquad & \text{in } \partial^+ S_T. \end{aligned}$$

Then the optimal stopping iterate $\tilde{v}^n$ defined in (3.26) for $n \geq 1$ is the unique (strong) viscosity solution of (3.27).

Now let us prove the equivalence of the two iterations (3.24) and (3.26). For this, we include in our set of assumptions the impulse control specific conditions (V1) and (V2), and require only the continuity of the iteration start v^0; note that the continuity of v^0 implies the continuity of g.

Proposition 3.4.4. *Let Assumptions 2.2.1, 2.2.3, 2.2.4, as well as (E4) be satisfied. Let $v^n, \tilde{v}^n \in \mathcal{PB}_p([0,T]\times\mathbb{R}^d)$, and v^0 be continuous. Further assume for any point $(t_0,x_0) \in \partial^* S_T$ and sequence $S_T \ni (t_k,x_k) \to (t_0,x_0)$ that $\limsup_{k\to\infty} \tilde{v}^n(t_k,x_k) \leq \max(\tilde{v}^{n-1}, \mathcal{M}\tilde{v}^{n-1})(t_0,x_0)$, and assume that there is a strict supersolution w as in Prop. 3.4.3.*
Then the iterations (3.24) and (3.26) are equivalent, i.e., $v^n = \tilde{v}^n$ for all $n \geq 1$. In particular, v^n is continuous on $[0,T]\times\mathbb{R}^d$ and a strong viscosity solution of (3.25).

Proof: We show that v^n is a (strong) viscosity solution of (3.27). The desired equality follows because $\tilde{v}^n$ is the unique viscosity solution of (3.27).

First note that under the assumptions, v^n is a weak viscosity solution of (3.25). It is immediate from the definition that $\tilde{v}^n \geq v^n$. Furthermore, $v^0 = \tilde{v}^0$ is continuous; for the induction step for $n \geq 1$, we can thus assume that $\tilde{v}^{n-1} = v^{n-1}$ and that v^{n-1}, $\mathcal{M}v^{n-1}$ are continuous (the latter by Lemma 2.4.3).

1. We start by considering the values on the nonlocal boundary $\partial^+ S_T$: By the increasingness property $v^n \geq v^{n-1}$, we have $\mathcal{M}v^{n-1} \leq \max(g, \mathcal{M}v^{n-1}) \leq \max(v^{n-1}, \mathcal{M}v^{n-1})$. On the other hand, by definition and increasingness we have $v^{n-1} \leq v^n = \max(g, \mathcal{M}v^{n-1})$ in $\partial^+ S_T$, thus

$$\max(v^{n-1}, \mathcal{M}v^{n-1}) \leq v^n = \max(g, \mathcal{M}v^{n-1}) \leq \max(v^{n-1}, \mathcal{M}v^{n-1}), \tag{3.28}$$

and hence the boundary conditions are equal.

2. Now we show that v^n is a (strong) viscosity solution of (3.27). The boundary conditions are satisfied by our arguments in **1** except on $\partial^* S_T$, where they may be affected by taking the lower or upper semicontinuous envelope. We know that by the increasingness property of v^n and by the supersolution inequality $v^n \geq \mathcal{M}v^{n-1}$ in $[0,T] \times \mathbb{R}^d$, that $(v^n)^* \geq (v^n)_* \geq \max(v^{n-1}, \mathcal{M}v^{n-1})$ in $[0,T] \times \mathbb{R}^d$. The reverse inequality holds on $\partial^* S_T$ because $v^n \leq \tilde{v}^n$ and assumption (E3) on $\tilde{v}^n$.

Fix some $(t_0, x_0) \in S_T$. To show the subsolution properties, let $\varphi \in \mathcal{PB} \cap C^{1,2}([0,T) \times \mathbb{R}^d)$ with $\varphi(t_0,x_0) = (v^n)^*(t_0,x_0)$, $\varphi \geq (v^n)^*$ on $[0,T) \times \mathbb{R}^d$. We can apply for this φ the viscosity subsolution property of v^n: in (t_0,x_0) either $-\sup_{\beta \in B}\{\varphi_t + \mathcal{L}^\beta \varphi + f^\beta\} \leq 0$, or $(v^n)^* \leq \mathcal{M}v^{n-1} \leq \max(v^{n-1}, \mathcal{M}v^{n-1})$.

The supersolution property is proved similarly, using the increasingness of the sequence (v^n), and thus $(v^n)_* \geq v^{n-1}$. □

It is not necessary to prove that $\tilde{v}^n$ is a weak viscosity solution of (3.25) for our purposes. However, the proof helps us to understand why both iterations are equivalent:

Proof of Prop. 3.4.4, part 3: We show that $\tilde{v}^n$ is a weak viscosity solution of (3.25). The boundary conditions hold by (3.28) and continuity of $\tilde{v}^n$. Let $(t_0,x_0) \in S_T$. To show the subsolution properties, assume we are given $\varphi \in \mathcal{PB} \cap C^{1,2}([0,T) \times \mathbb{R}^d)$ with $\varphi(t_0,x_0) = \tilde{v}^n(t_0,x_0)$, $\varphi \geq \tilde{v}^n$ on $[0,T) \times \mathbb{R}^d$. We have to show that

$$\min(-\sup_{\beta \in B}\{\varphi_t + \mathcal{L}^\beta \varphi + f^\beta\}, \tilde{v}^n - \mathcal{M}v^{n-1}) \leq 0$$

in (t_0,x_0). If $\tilde{v}^n - \mathcal{M}v^{n-1} \leq 0$, then there is nothing to prove. Let us assume that the reverse is the case; then either $\tilde{v}^n - \max(v^{n-1}, \mathcal{M}v^{n-1}) > 0$ and thus $-\sup_{\beta \in B}\{\varphi_t + \mathcal{L}^\beta \varphi + f^\beta\} \leq 0$ by the viscosity property of $\tilde{v}^n$, or $\tilde{v}^n - \max(v^{n-1}, \mathcal{M}v^{n-1}) \leq 0$. In the latter case, then $\tilde{v}^n(t_0,x_0) = v^{n-1}(t_0,x_0)$ by the increasingness property.

But this means $\varphi(t_0,x_0) = v^{n-1}(t_0,x_0)$ and $\varphi \geq \tilde{v}^n \geq v^{n-1}$ on $[0,T) \times \mathbb{R}^d$, and we can apply the induction hypothesis in (t_0,x_0): either $-\sup_{\beta \in B}\{\varphi_t + \mathcal{L}^\beta \varphi + f^\beta\} \leq 0$, or

$$0 \geq v^{n-1} - \mathcal{M}v^{n-2} \geq \tilde{v}^n - \mathcal{M}v^{n-1},$$

where the last inequality holds by monotonicity of $\mathcal{M}$ and increasingness of (v^n).

It is straightforward to check the supersolution properties of $\tilde{v}^n$ in (3.25). □

The equivalence of (3.24) and (3.26) shows that if less than n impulses are optimal, then there are two equally optimal strategies in the n-th iteration: (a) Do the impulses in the outer iterations, and none in the inner ones (form (3.24) and (3.26)); (b) do the impulses in the inner iterations, and none in the outer ones (only form (3.26)).

Remark 3.4.1. We haved used the equivalence of the two optimal stopping iterations to prove that v^n from (3.24) is continuous. Alternatively, by adapting the HJBVI existence and uniqueness results in §3.3, one could also directly prove that v^n is continuous if one requires its continuity at the parabolic boundary $\partial^* S_T$.

3.4.2 Convergence of iterated optimal stopping

To prove convergence of the optimal stoppping iteration, we employ some techniques that have been successfully used for diffusions already in Bensoussan and Lions [17] and seem to be due to

Hanouzet and Joly [60]; our approach borrows ideas from the more recent work Morimoto [86], who deals with combined stochastic control and stopping of a particular diffusion. Our proof of the contraction-like properties of the optimal stopping operator Q highlights the essential role that the strict supersolution of the HJBQVI of impulse control plays.

As standing assumption for this subsection, we take all requirements for granted that are needed for viscosity solution existence and uniqueness of the HJBQVI (2.10) (see §2.2). In particular, let Assumptions 2.2.1-2.2.4 be satisfied, and the space of polynomially bounded functions $\mathcal{PB}$ be fixed. The value function of combined stochastic and impulse control (denoted in this subsection by v) is assumed to be in $\mathcal{PB}_p$, and there exist a nonnegative smooth strict HJBQVI supersolution w increasing faster than $|x|^p$ for $|x| \to \infty$.

Because adding positive constants to w maintains the strict supersolution properties of w, we may without loss of generality assume that $v \leq w - \kappa$ for $\kappa > 0$ and that $w > 0$, thereby simplifying the calculations. Furthermore we will assume in this subsection that $v^0 \in \mathcal{PB}_p$ and continuous, and that v^n (or, equivalently, $\tilde{v}^n$) is continuous at the boundary (see below).

Parabolic case

We first consider the time-dependent parabolic case starting in $(t, x) \in [0, T] \times \mathbb{R}^d$ with horizon $T > t$. Define the optimal stopping operator Q by

$$Qu(t,x) := \sup_{\beta\in\mathcal{B},\tau\in\mathcal{T}} \mathbb{E}^{(t,x)} \left[\int_t^\tau f(s, X_s^\beta, \beta_s)ds + \mathcal{M}u(\tau, X_\tau^\beta)1_{\tau<\tau_S^T} + \max\left(g, \mathcal{M}u\right)(\tau, X_\tau^\beta)1_{\tau=\tau_S^T} \right],$$

where the supremum is taken over all admissible stochastic controls $\beta \in \mathcal{B}$ and all stopping times $\tau \in \mathcal{T}$, i.e., $t \leq \tau \leq \tau_S^T = \tau_S \wedge T$. With this notation, the optimal stopping iteration reads

$$v^n = Qv^{n-1}, \quad n \geq 1, \qquad v^0(t,x) = \sup_{\beta\in\mathcal{B}} \mathbb{E}^{(t,x)} \left[\int_t^{\tau_S^T} f(s, X_s^\beta, \beta_s)ds + g(\tau_S^T, X_{\tau_S^T}^\beta) \right]. \tag{3.29}$$

For the impulse control value function v from (2.7), we know that $Qv = v$ if $v \in \mathcal{PB}_p([0,T] \times \mathbb{R}^d)$. Indeed, by the HJB(Q)VI viscosity characteristics, any continuous solution u of the HJBQVI (2.10) solves an HJBVI with obstacle $\mathcal{M}u$ in S_T and $\max(g, \mathcal{M}u)$ on $\partial^+ S_T$; if $u \in \mathcal{PB}_p$, then $u = Qu$. It is clear that $v^0 \leq v$ because no impulse is an admissible impulse control strategy. Because Q is monotone by Prop. 3.4.1, we thus have for $n \geq 1$

$$v^0 \leq Qv^{n-1} = v^n \leq Q^n v \leq v \leq w, \tag{3.30}$$

and the same for $\tilde{v}^n$. We assume $v^0, v \in \mathcal{PB}_p([0,T] \times \mathbb{R}^d)$, thus also $v^n \in \mathcal{PB}_p([0,T] \times \mathbb{R}^d)$. The strict HJBQVI supersolution $w \in \mathcal{PB} \cap C^{1,2}([0,T] \times \mathbb{R}^d)$ satisfies in the parabolic case for some $\kappa > 0$

$$\begin{aligned} \min(-\sup_{\beta\in B}\{w_t + \mathcal{L}^\beta w + f^\beta\}, w - \mathcal{M}w) &\geq \kappa \qquad \text{in } S_T \\ \min(w - g, w - \mathcal{M}w) &\geq \kappa \qquad \text{in } \partial^+ S_T. \end{aligned}$$

Because by (3.30) we have $w - \kappa \geq \tilde{v}^{n-1}$ and $\mathcal{M}w \geq \mathcal{M}\tilde{v}^{n-1}$, the HJBQVI strict supersolution w is also a strict supersolution for the HJBVI (3.27).

We assume that v^0 is continuous and for any $n \geq 1$ and any point $(t_0, x_0) \in \partial^* S_T$ and sequence $S_T \ni (t_k, x_k) \to (t_0, x_0)$ that $\limsup_{k\to\infty} \tilde{v}^n(t_k, x_k) \leq \max(\tilde{v}^{n-1}, \mathcal{M}\tilde{v}^{n-1})(t_0, x_0)$. Which regularity conditions at the boundary are sufficient for this was already discussed at the end of §2.4.1.

Under the above assumptions, Propositions 3.4.2, 3.4.3 and 3.4.4 hold and thus v^n and $\tilde{v}^n$ are equal and continuous viscosity solutions of the corresponding variational inequalities. We start our convergence proof with a lemma stating some contraction-like properties of Q:

Lemma 3.4.5. *Assume f and $\mathcal{M}w$ are bounded from below. Let $u, \tilde{u} \geq v^0$, and $(Q\tilde{u})^-$ locally bounded. If $u - \tilde{u} \leq \lambda(w - \tilde{u})$ for some $\lambda \in [0,1]$, then for each compact C there is a constant $\mu \in (0,1]$ such that on C*

$$Qu - Q\tilde{u} \leq \lambda(1-\mu)(w - Q\tilde{u}).$$

Proof: By Prop. 3.4.1, Q is a monotone operator. Furthermore, Q is convex, which follows immediately from the corresponding property of the impulse intervention operator $\mathcal{M}$ (Lemma 2.5.7), and thus

$$Qu \leq Q((1-\lambda)\tilde{u} + \lambda w) \leq (1-\lambda)Q\tilde{u} + \lambda Qw.$$

From this we obtain the inequality $Qu - Q\tilde{u} \leq \lambda(Qw - Q\tilde{u})$. It thus remains to prove

$$Qw \leq (1-\mu)w + \mu Q\tilde{u}$$

for a suitable constant $\mu > 0$; note that if $Q\tilde{u} \geq 0$, then the second summand can be dropped. By the strict supersolution properties of w and Dynkin's formula, we have for any stopping time $\tau \in \mathcal{T}$ and $\tau_R := \tau \wedge \inf\{s \geq t : |X_s^{\beta,t,x} - x| \geq R\} \wedge R$ for $R > t$:

$$\begin{aligned}
& \mathbb{E}^{(t,x)}\left[\int_t^{\tau_R} f(s, X_s^\beta, \beta_s)ds + \mathcal{M}w(\tau_R, X_{\tau_R}^\beta)1_{\tau_R < \tau_S^T} + \max\left(g, \mathcal{M}w\right)(\tau_R, X_{\tau_R}^\beta)1_{\tau_R = \tau_S^T}\right] \\
\leq\ & \mathbb{E}^{(t,x)}\left[\int_t^{\tau_R} f(s, X_s^\beta, \beta_s)ds + w(\tau_R, X_{\tau_R}^\beta)\right] - \kappa \\
=\ & w(t,x) + \mathbb{E}^{(t,x)}\left[\int_t^{\tau_R} f(s, X_s^\beta, \beta_s) + w_t(s, X_s^\beta) + \mathcal{L}^{\beta_s}w(s, X_s^\beta)\, ds\right] - \kappa \qquad (3.31) \\
\leq\ & w(t,x) - \kappa\, \mathbb{E}^{(t,x)}\left[(\tau_R - t) + 1\right] \leq w(t,x) - \kappa.
\end{aligned}$$

We apply Fatou's lemma for $R \to \infty$, and choose $\mu \in (0,1]$ such that $\mu(w + (Q\tilde{u})^-) \leq \kappa$; this is achieved by

$$\mu = \mu_C := \min\left(1, \frac{\kappa}{\|w + (Q\tilde{u})^-\|_{\infty,C}}\right),$$

where $\|\cdot\|_{\infty,C}$ is the supremum norm on C, and thus the desired inequality holds. □

The additional assumptions needed for the application of Fatou's lemma are weak: $\mathcal{M}w$ is bounded from below already if K is bounded from below. Alternatively, a version of the dominated convergence theorem can be applied if $\int_t^{\tau_R} f(s, X_s^\beta, \beta_s)ds + \mathcal{M}w(\tau_R, X_{\tau_R}^\beta)$ is bounded from below by some L^1-integrable random variable independent of R. Note that in principle, also a viscosity solution comparison result could be applied.

Theorem 3.4.6 (Convergence of iterated optimal stopping)**.** *Assume f and $\mathcal{M}w$ are bounded from below. Then the iterated optimal stopping sequence v^n defined by* (3.24) *converges to the value function v in* (2.7) *of combined stochastic and impulse control, uniformly on each compact.*

Proof: We first note that the sequence $(v^n)_{n\geq 0}$ is monotonically increasing (Prop. 3.4.1), and we have

$$v^0 \leq v^{n-1} \leq v^n \leq w.$$

This yields immediately $0 \leq v^1 - v^0 \leq w - v^0$ and hence, by an application of Lemma 3.4.5, $v^2 - v^1 = Qv^1 - Qv^0 \leq (1-\mu)(w - v^1)$ on a fixed compact C and some $\mu \in (0,1]$. By iteration, we obtain

$$0 \leq v^{n+1} - v^n \leq (1-\mu)^n (w - v^n) \leq (1-\mu)^n (w - v^0),$$

which proves that v^n converges uniformly on C to some function $\overline{v}$ ($\in \mathcal{PB}_p$). Due to the continuity of $\mathcal{M}$ (Lemma 2.4.3 (vi)), we also have $\mathcal{M}v^n \to \mathcal{M}\overline{v}$ uniformly on compacts.

Because v^n is a continuous (strong) viscosity solution of the HJBVI (3.25) by Prop. 3.4.4, we can apply the stability result (Th. 3.3.8) and conclude that $\overline{v}$ is a continuous viscosity solution of

$$\begin{aligned} \min(-\sup_{\beta\in B}\{u_t + \mathcal{L}^\beta u + f^\beta\}, u - \mathcal{M}\overline{v}) &= 0 \qquad \text{in } S_T \\ u - \max(g, \mathcal{M}\overline{v}) &= 0 \qquad \text{in } \partial^+ S_T. \end{aligned}$$

Noting that $u - \max(g, \mathcal{M}\overline{v}) = \min(u-g, u-\mathcal{M}\overline{v})$, we can conclude that $\overline{v}$ is a (strong) viscosity solution of the HJBQVI (2.10) of impulse control, and by uniqueness (Th. 2.2.2), $\overline{v}$ is equal to the value function v of combined stochastic and impulse control. $\square$

The proof of Theorem 3.4.6 gives us an explicit estimate of the convergence speed of iterated optimal stopping. Roughly, the convergence speed is determined by κ divided by the strict supersolution w. From the discussion of the assumptions in §2.2, we recall that κ in the parabolic case was mainly dependent on the fixed costs; the higher the fixed costs of an impulse, the larger κ can be chosen. Thus as intuition suggests, iterated optimal stopping is faster for higher fixed costs.

Elliptic case

The elliptic, time-independent case works very similar to the parabolic, time-dependent case. Define the optimal stopping operator Q by

$$Qu(x) := \sup_{\substack{\beta\in\mathcal{B}\\ \tau\in\mathcal{T}}} \mathbb{E}^x \left[\int_0^\tau e^{-\rho s} f(X_s^\beta, \beta_s) ds + e^{-\rho\tau} \left(\mathcal{M}u(X_\tau^\beta) 1_{\tau<\tau_S} + \max\left(g, \mathcal{M}u\right)(X_\tau^\beta) 1_{\tau=\tau_S} \right) \right],$$

where the supremum is taken over all admissible stochastic controls $\beta \in \mathcal{B}$ and all stopping times $\tau \leq \tau_S$, the first exit time from the open set S. With this notation, the optimal stopping iteration reads

$$v^n = Qv^{n-1}, \quad n \geq 1, \qquad v^0(x) = \sup_{\beta\in\mathcal{B}} \mathbb{E}^{(x)} \left[\int_0^{\tau_S} e^{-\rho s} f(X_s^\beta, \beta_s) ds + e^{-\rho\tau_S} g(X_{\tau_S}^\beta) \right]. \tag{3.32}$$

All assumptions from the parabolic part are needed here as well, in particular the continuity of v^0, and of $\tilde{v}^n$ at the boundary ∂S. We again assume $v^0, v \in \mathcal{PB}_p(\mathbb{R}^d)$, from which follows that $v^n \in \mathcal{PB}_p(\mathbb{R}^d)$ for all $n \geq 1$. The parabolic results from §3.4.1 hold correspondingly for the elliptic case. The strict HJBQVI supersolution $w \in \mathcal{PB} \cap C^2(\mathbb{R}^d)$, with $w \geq v + \kappa$, $w > 0$ satisfies in the elliptic case for a $\kappa > 0$

$$\min(-\sup_{\beta \in B}\{-\rho w + \mathcal{L}^\beta w + f^\beta\}, w - \mathcal{M}w) \geq \kappa \qquad \text{in } S$$
$$\min(w - g, w - \mathcal{M}w) \geq \kappa \qquad \text{in } \mathbb{R}^d \setminus S,$$

where $\mathcal{L}^\beta$ is the elliptic integro-differential operator defined in (2.16), and $\mathcal{M}$ is the impulse intervention operator. By the same arguments as above, the HJBQVI strict supersolution w is also a strict supersolution of the elliptic version of HJBVI (3.27).

The proof of Lemma 3.4.5 changes only in that Dynkin's formula is applied to $\mathbb{E}^x[e^{-\rho s} w(X_s)]$, and consequently $\sup_{\beta \in B}\{-\rho w + \mathcal{L}^\beta w + f^\beta\} \leq 0$ is needed.

Theorem 3.4.7 (Convergence of iterated optimal stopping, time-independent case)**.** *Assume f and $\mathcal{M}w$ are bounded from below. Then the iterated optimal stopping sequence v^n defined by* (3.32) *converges to the value function v in* (2.32) *of combined stochastic and impulse control, uniformly on each compact.*

3.4.3 Alternative proof of HJBQVI existence and uniqueness

By modifying slightly the convergence proof for iterated optimal stopping in §3.4.2, we can obtain a viscosity solution existence and uniqueness result for HJBQVI which holds true under slightly weaker conditions than in Chapter 2. We state this result only in the parabolic case.

The condition to be weakened is (E3), the continuity at the boundary $\partial^* S_T$. We discussed already at the end of §2.4.1 that (E1*), the regularity of the process X at $\partial^* S_T$, can be a substitute. (E1*) on v^0 replaces the lim inf part of (E3), whereas (E1*) on v^n for $n \geq 1$ replaces the lim sup part of (E3) and allows us to get rid of the distracting $v > \mathcal{M}v$ side condition. This side condition is not necessary anymore because in the construction via iterated optimal stopping, we work on the space of continuous functions, and thus $\{(t,x) : v(t,x) > \mathcal{M}v(t,x)\}$ is open (compare Prop. 2.4.4).

We assume here the integrability condition in Prop. 2.4.4. Then by (E1*) (without impulse control) and continuity of f and g, the start iterate v^0 is continuous. The condition $\limsup_{k\to\infty} \tilde{v}^n(t_k, x_k) \leq \max(\tilde{v}^{n-1}, \mathcal{M}\tilde{v}^{n-1})(t_0, x_0)$ holds as well by (E1*) (with impulse control replaced by stopping): In the proof of Prop. 2.4.4, either X^{β_n, t_n, x_n} leaves S, or it is stopped and the claim holds by continuity of $\max(\tilde{v}^{n-1}, \mathcal{M}\tilde{v}^{n-1})$.

Because $Qv \leq v$ cannot be shown using the HJBQVI viscosity characteristics as in §3.4.2, we have to assume that $Qv \leq v$ holds (e.g., by dynamic programming). Alternatively, one can directly require all v^n to be in $\mathcal{PB}_p([0,T] \times \mathbb{R}^d)$.

We summarize the result in the following theorem, where v denotes again the value function of combined stochastic and impulse control.

Theorem 3.4.8 (Viscosity existence and uniqueness 2 for parabolic HJBQVI)**.** *Let Assumptions 2.2.1, 2.2.3, 2.2.4, and further (E2), (E1*), (E4) be satisfied. Assume furthermore that $v^0, v \in \mathcal{PB}_p([0,T] \times \mathbb{R}^d)$, and that there is a nonnegative $w \in \mathcal{PB} \cap C^{1,2}([0,T] \times \mathbb{R}^d)$ with $w(t,x)/|x|^p \to \infty$ for $|x| \to \infty$ (uniformly in t) and a constant $\kappa > 0$ such that*

$$\min(-\sup_{\beta \in B}\{w_t + \mathcal{L}^\beta w + f^\beta\}, w - \mathcal{M}w) \geq \kappa \qquad \textit{in } S_T$$
$$\min(w - g, w - \mathcal{M}w) \geq \kappa \qquad \textit{in } \partial^+ S_T.$$

Let f and $\mathcal{M}w$ be bounded from below. Then there is a viscosity solution of the parabolic HJBQVI (2.10) continuous on $[0,T] \times \mathbb{R}^d$, which is unique in $\mathcal{PB}_p \cap C([0,T] \times \mathbb{R}^d)$. This solution is the locally uniform limit of the optimal stopping iterates v^n defined by (3.24).

Proof: The existence was already proved in Th. 3.4.6, using wlog $w \geq v + \kappa$. The limit $\overline{v}$ of iterated optimal stopping is a continuous viscosity solution of the HJBQVI (2.10).

For the uniqueness, first observe that by the HJBVI viscosity characteristics (Th. 3.3.1), any continuous viscosity solution $u \in \mathcal{PB}_p$ of the HJBQVI (2.10) satisfies $u = Qu$ for the parabolic optimal stopping operator Q of §3.4.2. Assume $u_1, u_2 \in \mathcal{PB}_p$ with $u_1, u_2 \leq w$ are two continuous viscosity solutions of (2.10), then $u_1 - u_2 \leq w - u_2$, and thus by Lemma 3.4.5

$$u_1 - u_2 = Qu_1 - Qu_2 \leq (1-\mu)(w - Qu_2) = (1-\mu)(w - u_2).$$

Iterating this procedure infinitely often, we obtain $u_1 \leq u_2$ (and vice versa). The restriction $\leq w$ can be omitted, because we can wlog add positive constants to w, and w increases faster than $|x|^p$ for $|x| \to \infty$. □

3.4.4 Obtaining an optimal strategy

We investigate in this subsection how to derive from the value functions of iterated optimal stopping an approximately optimal strategy for an impulse control problem.

There is a natural impulse control strategy which is inherent in the definition of the optimal stopping iteration (3.24): No impulse is contained in v^0. To get the expected payoff v^1, a possible strategy is to jump in points (t,x) where $v^1(t,x) = \mathcal{M}v^0(t,x)$ and to choose as impulse size a maximizer in $\mathcal{M}v^0(t,x)$; note that $v^j \geq \mathcal{M}v^{j-1}$ always holds. For v^2, the controller should jump for the first time if $v^2(t,x) = \mathcal{M}v^1(t,x)$, and for the second and last time if $v^1(t,x) = \mathcal{M}v^0(t,x)$. In the n-th iteration, the strategy would be:

> Do the first impulse according to the rule (J-n) "Jump if $v^n = \mathcal{M}v^{n-1}$", and the n-th (last) impulse according to (J-1) "Jump if $v^1 = \mathcal{M}v^0$".

It is implicitly understood that one always chooses the corresponding maximizer in $\mathcal{M}v^{n-j}$ as impulse size (assuming it exists). The corresponding impulse times in the combined stochastic and impulse control problem are

$$\begin{aligned} \tau_1 &= \inf\{s \geq t : v^n(s, X_s^{\beta,t,x}) = \mathcal{M}v^{n-1}(s, X_s^{\beta,t,x})\} \\ \tau_j &= \inf\{s \geq \tau_{j-1} : v^{n-j+1}(s, X_s^{\alpha_{j-1},t,x}) = \mathcal{M}v^{n-j}(s, X_s^{\alpha_{j-1},t,x})\}, \qquad j = 2, \ldots, n, \end{aligned} \tag{3.33}$$

where X^{α_j} is the process controlled by the first j impulses, and by suitable stochastic control. The (τ_j) are exit times only if we add the number of effected impulses as additional component to the process X; this additional component jumps by 1 at the time of an impulse. Note that by their definition, the stopping times $(\tau_j)_{j\geq 1}$ do no necessarily occur before τ_S^T.

Why should one choose the impulse strategy (3.33)? First, the optimality of the above strategies in the combined stochastic control and stopping problem has to be considered. For v^0, there is nothing to show. For v^n with $n \geq 1$, there are several possible cases to investigate:

1. $(t,x) \in S_T$, $v^n(t,x) > \mathcal{M}v^{n-1}(t,x)$: Stopping is not optimal, there is a pair $(\beta, \tau) \in \mathcal{B} \times \mathcal{T}$ with expected payoff $> \mathcal{M}v^{n-1}(t,x)$.

2. $(t,x) \in S_T$, $v^n(t,x) = \mathcal{M}v^{n-1}(t,x)$: Stopping is optimal, $v^n(t,x)$ is the immediate payoff.
3. $(t,x) \in \partial^+ S_T$: Stopping is obligatory. If $v^n(t,x) > \mathcal{M}v^{n-1}(t,x)$, then $g(t,x)$ will be chosen, else the impulse option $\mathcal{M}v^{n-1}(t,x)$ will be chosen.

So in S_T, *immediate stopping is optimal if and only if* $v^n(t,x) = \mathcal{M}v^{n-1}(t,x)$.

With the above optimal stopping strategies, we can motivate the impulse control strategy (3.33). We still have to argue why, together with a suitable stochastic control, (3.33) in an impulse control problem approximates v^n. For simplicity, consider only v^1, and assume that $g > \mathcal{M}v^0$ on $\partial^+ S_T$. For some small enough $\varepsilon > 0$, choose a stochastic control and stopping strategy $(\beta^\varepsilon, \tau_1)$ which is ε-optimal in v^1; note that here $\tau_1 < \tau_S^T$. Choose further a $(\tau_1, X_{\tau_1}^{\beta^\varepsilon})$-measurable impulse size ζ^ε, and $\tilde{\beta}^\varepsilon \in \mathcal{B}(\tau_1, \Gamma(\tau_1, X_{\tau_1}^{\beta^\varepsilon}, \zeta^\varepsilon))$ such that $(\zeta^\varepsilon, \tilde{\beta}^\varepsilon)$ are ε-optimal for $\mathcal{M}v^0(\tau_1, X_{\tau_1}^{\beta^\varepsilon})$.[2] Denote again by β^ε the extension of β^ε to $[t, \tau_S^T]$ via $\tilde{\beta}^\varepsilon$, and assume $\beta^\varepsilon \in \mathcal{B}(t,x)$. Then apply $\alpha^\varepsilon = (\beta^\varepsilon, \tau_1, \zeta^\varepsilon)$ as combined stochastic and impulse control with one impulse. The expected payoff is

$$
\begin{aligned}
& \mathbb{E}^{(t,x)}\left[\int_t^{\tau_S^T} f(s, X_s^{\alpha^\varepsilon}, \beta_s^\varepsilon)ds + g(\tau_S^T, X_{\tau_S^T}^{\alpha^\varepsilon}) + K(\tau_1, X_{\tau_1}^{\beta^\varepsilon}, \zeta^\varepsilon)\right] \\
= \; & \mathbb{E}^{(t,x)}\Bigg[\int_t^{\tau_1} f(s, X_s^{\beta^\varepsilon}, \beta_s^\varepsilon)ds + g(\tau_1, X_{\tau_1}^{\beta^\varepsilon})1_{\tau_1 = \tau_S^T} + \Big\{K(\tau, X_\tau^{\beta^\varepsilon}, \zeta^\varepsilon) \\
& + \mathbb{E}^{(\tau_1, \Gamma(\tau_1, X_{\tau_1}^{\beta^\varepsilon}, \zeta^\varepsilon))}\left[\int_{\tau_1}^{\tau_S^T} f(s, X_s^{\beta^\varepsilon}, \beta_s^\varepsilon)ds + g(\tau_S^T, X_{\tau_S^T}^{\beta^\varepsilon})\right]\Big\} 1_{\tau_1 < \tau_S^T}\Bigg] \\
\geq \; & \mathbb{E}^{(t,x)}\left[\int_t^{\tau_1} f(s, X_s^{\beta^\varepsilon}, \beta_s^\varepsilon)ds + g(\tau_1, X_{\tau_1}^{\beta^\varepsilon})1_{\tau_1 = \tau_S^T} + \mathcal{M}v^0(\tau_1, X_{\tau_1}^{\beta^\varepsilon})1_{\tau_1 < \tau_S^T}\right] - \varepsilon \\
\geq \; & v^1(t,x) - 2\varepsilon,
\end{aligned}
$$

where in the first equality, we have switched from stochastic-and-impulse-controlled X^α to a simple stochastic process X^β following an SDE controlled by β. Thus we can get arbitrarily close to v^n by choosing a combined stochastic and impulse control with at most n impulses of the type (3.33).

More generally, again assuming that $g > \mathcal{M}v^0$ on $\partial^+ S_T$ and that everything is well behaved, we have by similar, heuristic arguments

$$
\begin{aligned}
v^1(t,x) = & \sup_{\beta \in \mathcal{B}(t,x), \tau \in \mathcal{T}(t,x)} \mathbb{E}^{(t,x)}\left[\int_t^\tau f(s, X_s^\beta, \beta_s)ds + \mathcal{M}v^0(\tau, X_\tau^\beta)1_{\tau < \tau_S^T} + g(\tau, X_\tau^\beta)1_{\tau = \tau_S^T}\right] \\
= & \sup_{\beta \in \mathcal{B}(t,x), \tau \in \mathcal{T}(t,x)} \mathbb{E}^{(t,x)}\Bigg[\int_t^\tau f(s, X_s^\beta, \beta_s)ds + g(\tau, X_\tau^\beta)1_{\tau = \tau_S^T} + \sup_{\zeta \in Z(\tau, X_\tau^\beta)} \Big\{K(\tau, X_\tau^\beta, \zeta) \\
& + \sup_{\beta \in \mathcal{B}(\tau, \Gamma(\tau, X_\tau^\beta, \zeta))} \mathbb{E}^{(\tau, \Gamma(\tau, X_\tau^\beta, \zeta))}\left[\int_\tau^{\tau_S^T} f(s, X_s^\beta, \beta_s)ds + g(\tau_S^T, X_{\tau_S^T}^\beta)\right]\Big\} 1_{\tau < \tau_S^T}\Bigg] \\
= & \sup_{\alpha = (\beta, \tau, \zeta) \in \mathcal{A}(t,x)} \mathbb{E}^{(t,x)}\left[\int_t^\tau f(s, X_s^\alpha, \beta_s)ds + g(\tau_S^T, X_{\tau_S^T}^\alpha) + K(\tau, X_\tau^\beta, \zeta)\right],
\end{aligned}
\tag{3.34}
$$

where in the last step we have drawn out the pair (τ, ζ), and replaced the stochastic process X^β following a controlled SDE by a stochastic-and-impulse-controlled X^α. The sup can be drawn out

[2] This uses the (strong) Markov property.

basically because optimization over different time intervals is independent. The interpretation of (3.34) in the general case for $n \geq 1$ is that the strategy (3.33) (together with a suitable stochastic control) is as optimal as if the true optimal strategy were capped after at most n impulses; for more rigorous results in this direction, at least in the infinite-horizon case, we refer to Øksendal and Sulem [93].

Alternative strategy. If we are actually allowed to effect infinitely many impulses, we can prove that we can do as least as good as v^n by employing the strategy:

Do all impulses according to the rule (J-n) "Jump if $v^n = \mathcal{M}v^{n-1}$".

This can be written in impulse time notation by

$$\begin{aligned} \hat{\tau}_1 &= \inf\{s \geq t : v^n(s, X_s^{\beta,t,x}) = \mathcal{M}v^{n-1}(s, X_s^{\beta,t,x})\} \\ \hat{\tau}_j &= \inf\{s \geq \hat{\tau}_{j-1} : v^n(s, X_s^{\hat{\alpha}_{j-1},t,x}) = \mathcal{M}v^{n-1}(s, X_s^{\hat{\alpha}_{j-1},t,x})\}, \qquad j = 2, \dots, n, \end{aligned} \tag{3.35}$$

where $X^{\hat{\alpha}_{j-1},t,x}$ has an analogous meaning as in (3.33). To understand the advantage of this second strategy, consider the case $n = 1$: Under the original strategy (3.33), the controller will do only one jump. Under the modified strategy (3.35), the controller is to his surprise allowed to effect yet another impulse after his first one — this means his (imagined) payoff after the first impulse is not v^0, but rather v^1! Of course, this works only because our system is Markovian and thus "memoryless".

Proposition 3.4.9. *For $n, k \in \mathbb{N}$, define by $\tilde{v}^{n,k}$ the expected payoff if we apply the rule (J-n) k times, and then subsequently (J-(n-1)), ..., (J-1) as in* (3.33) *together with an optimal $\beta \in \mathcal{B}$; in other words, set $\tilde{v}^{n,1} := v^n$, and for $k \geq 2$ with the notation $\hat{\tau} := \hat{\tau}_1 \wedge \tau_S^T$:*

$$\begin{aligned} \tilde{v}^{n,k}(t,x) := \sup_{\beta \in \mathcal{B}(t,x)} \mathbb{E}^{(t,x)} \bigg[\int_t^{\hat{\tau}} & f(s, X_s^\beta, \beta_s) ds + \mathcal{M}\tilde{v}^{n,k-1}(\hat{\tau}, X_{\hat{\tau}}^\beta) 1_{\hat{\tau} < \tau_S^T} \\ & + \max(g, \mathcal{M}\tilde{v}^{n,k-1})(\hat{\tau}, X_{\hat{\tau}}^\beta) 1_{\hat{\tau} = \tau_S^T} \bigg]. \end{aligned}$$

Denote by v^n the value function obtained by the strategy (3.33). *Then for all $k \in \mathbb{N}$, $(t,x) \in [0,T] \times \mathbb{R}^d$,*

$$\tilde{v}^{n,k}(t,x) \geq v^n(t,x).$$

Proof: For $\varepsilon > 0$ small enough, choose a strategy $(\beta^\varepsilon, \hat{\tau}) \in \mathcal{B}(t,x) \times \mathcal{T}(t,x)$ which is ε-optimal in v^n, where $\hat{\tau} = \hat{\tau}_1 \wedge \tau_S^T$ is the first stopping time corresponding to (J-n) from (3.35). First note that by definition $\tilde{v}^{n,1} = v^n$. By induction over $k \geq 2$, we obtain for all $(t,x) \in [0,T] \times \mathbb{R}^d$

$$\begin{aligned} \tilde{v}^{n,k}(t,x) &\geq \mathbb{E}^{(t,x)} \bigg[\int_t^{\hat{\tau}} f(s, X_s^{\beta^\varepsilon}, \beta_s^\varepsilon) ds + \mathcal{M}\tilde{v}^{n,k-1}(\hat{\tau}, X_{\hat{\tau}}^{\beta^\varepsilon}) 1_{\hat{\tau} < \tau_S^T} \\ &\qquad\qquad + \max(g, \mathcal{M}\tilde{v}^{n,k-1})(\hat{\tau}, X_{\hat{\tau}}^{\beta^\varepsilon}) 1_{\hat{\tau} = \tau_S^T} \bigg] \\ &\geq \mathbb{E}^{(t,x)} \bigg[\int_t^{\hat{\tau}} f(s, X_s^{\beta^\varepsilon}, \beta_s^\varepsilon) ds + \mathcal{M}v^n(\hat{\tau}, X_{\hat{\tau}}^{\beta^\varepsilon}) 1_{\hat{\tau} < \tau_S^T} \\ &\qquad\qquad + \max(g, \mathcal{M}v^n)(\hat{\tau}, X_{\hat{\tau}}^{\beta^\varepsilon}) 1_{\hat{\tau} = \tau_S^T} \bigg] \end{aligned}$$

$$\begin{aligned}
&\geq \mathbb{E}^{(t,x)}\Big[\int_t^{\hat\tau} f(s, X_s^{\beta^\varepsilon}, \beta_s^\varepsilon)ds + \mathcal{M}v^{n-1}(\hat\tau, X_{\hat\tau}^{\beta^\varepsilon})1_{\hat\tau<\tau_S^T} \\
&\qquad\qquad\qquad\qquad\qquad + \max(g, \mathcal{M}v^{n-1})(\hat\tau, X_{\hat\tau}^{\beta^\varepsilon})1_{\hat\tau=\tau_S^T}\Big] \\
&\geq v^n(t,x) - \varepsilon,
\end{aligned}$$

where we have also used the monotonicity property of $\mathcal{M}$ (Lemma 2.4.3) and the increasingness of the sequence $(v^n)_n$ (Prop. 3.4.1). The result follows by $\varepsilon \to 0$. □

We can conclude from Prop. 3.4.9 that if with respect to a process impulse-controlled by rule (J-n), $\hat\tau_j \to \infty$ for $j \to \infty$, or

$$\mathbb{P}(\hat\tau_j > \tau_S^T) \to 1, \qquad j \to \infty,$$

then the rule (J-n) or (3.35) can be expected to perform better than (3.33).

3.5 Implementation of iterated optimal stopping

In this section, we present a possible numerical implementation of iterated optimal stopping, and discuss its convergence and several implementation details.

3.5.1 Numerical algorithm

We transform the terminal value problem by choosing an inverted time scale into an initial value problem on $S_0 := (0,T] \times S$ with nonlocal boundary $\partial^+ S_0 := \big(\{0\} \times \mathbb{R}^d\big) \cup \big((0,T] \times (\mathbb{R}^d \setminus S)\big)$. Then the iterated optimal stopping technique for the parabolic HJBQVI consists in finding the solution v^0 of

$$\begin{aligned}
-\sup_{\beta\in B}\{-u_t + \mathcal{L}^\beta u + f^\beta\} &= 0 \qquad \text{in } S_0 \\
u - g &= 0 \qquad \text{in } \partial^+ S_0,
\end{aligned} \tag{3.36}$$

and the solutions v^n for $n \geq 1$ of

$$\begin{aligned}
\min(-\sup_{\beta\in B}\{-u_t + \mathcal{L}^\beta u + f^\beta\}, u - \max(v^{n-1}, \mathcal{M}v^{n-1})) &= 0 \qquad \text{in } S_0 \\
u - \max(v^{n-1}, \mathcal{M}v^{n-1}) &= 0 \qquad \text{in } \partial^+ S_0.
\end{aligned} \tag{3.37}$$

It is not purpose of this thesis to give a general introduction to numerical solution of PDEs; for this, see for example Hundsdorfer and Verwer [62], Quarteroni et al. [102], or [108]. Nonetheless, we have to fix some notation. Therefore we consider the model problem $u_t - \sup_{\beta\in B}\{\beta u_{xx}\} = 0$, $u(0,\cdot) = g$ with Dirichlet boundary values of 0. Let us discretize B to B_h and the bounded $\overline{S} \subset \mathbb{R}$ to $(x_i)_{i=0,\dots,I}$, and assume that an equidistant grid with $x_i - x_{i-1} = h_x > 0$ is chosen. The model PDE on this grid becomes for $u_i(t) = u(t, x_i)$ a system of ODEs:

$$\sup_{\beta\in B_h}\left\{\frac{\beta}{(h_x)^2}\left(u_{i+1}(t) - 2u_i(t) + u_{i-1}(t)\right)\right\} = \dot u_i(t), \qquad i = 1, \dots, I-1,\ t > 0. \tag{3.38}$$

Here we set for simplicity $u_0(t) = u_I(t) = 0$ for all $t \in [0,T]$, and assume this implicitly also for the algorithm below. We further discretize also the time $[0,T]$ to $(t_j)_{j=0,\dots,J}$ with $t_j - t_{j-1} =$

$h_t > 0$, and introduce the notation $u_i^j = u(t_j, x_i)$ and $u^{(j)} = (u_i^j)_{i=1,\dots,I-1}$. Replacing $\dot{u}_i(t)$ by its finite difference approximation, and writing everything in matrix notation with the left hand side of (3.38) denoted by the finite difference matrix $A^\beta \in \mathbb{R}^{I\times I}$, we obtain

$$h_t \sup_{\beta\in B_h^I} \{A^\beta(\theta u^{(j)} + (1-\theta)u^{(j-1)})\} = u^{(j)} - u^{(j-1)}, \qquad j = 1,\dots,J. \tag{3.39}$$

Here we have also replaced $u(t)$ by the weighted average $\theta u^{(j)} + (1-\theta)u^{(j-1)}$ for $\theta \in [0,1]$, i.e., (3.39) uses mixed explicit-implicit timestepping (θ-method). The maximization for the vector $A^\beta u(t)$ is done separately for each dimension.

The advantage of a θ-method with $\theta > 0$ is improved stability compared to a pure explicit method (see, e.g., [62], [102], [108]). However, in this case (3.39) is a nonlinear equation in $u^{(j)}$, which makes it cumbersome to solve. Therefore in the algorithm below, the computation of the optimal $\hat{\beta}$ is done completely on the known $u^{(j-1)}$, resulting in a linear equation to solve for $u^{(j)}$.

We have now everything in place to state the iterated optimal stopping algorithm for the parabolic HJBQVI (3.1), where we assume $f = 0$:

Initialize data
Solve HJBVE $\longrightarrow v^0$:
    Set $u^{(0)} = g$ (initial condition)
    FOR $j = 1 : tn$
        Find $\hat{\beta}_i$ for each point x_i such that $(A^\beta)_{i*}u^{(j-1)}$ is maximal
        Solve $u^{(j)} - u^{(j-1)} = h_t A^{\hat{\beta}}(\theta u^{(j)} + (1-\theta)u^{(j-1)})$ for $u^{(j)}$
    END
    Set $v^0 = u$
FOR $n = 1 : N$
    Calculate $\mathcal{M}v^{n-1}$ for each point (t,x)
    Solve HJBVI with obstacle $\psi = \max(v^{n-1}, \mathcal{M}v^{n-1}) \longrightarrow v^n$:
        Set $u^{(0)} = \psi^{(0)}$ (initial condition)
        FOR $j = 1 : J$
            Find $\hat{\beta}_i$ for each point x_i such that $(A^\beta)_{i*}u^{(j-1)}$ is maximal
            Solve $u^{(j)} - u^{(j-1)} - h_t A^{\hat{\beta}}(\theta u^{(j)} + (1-\theta)u^{(j-1)}) \geq 0$, $u^{(j)} - \psi^{(j)} \geq 0$,
            $(u^{(j)} - u^{(j-1)} - h_t A^{\hat{\beta}}(\theta u^{(j)} + (1-\theta)u^{(j-1)})) \cdot (u^{(j)} - \psi^{(j)}) = 0$ for $u^{(j)}$
        END
        Set $v^n = u$
UNTIL convergence

The finite difference matrix A^β is tridiagonal for a PDE, but for a general PIDE, A^β can be a full matrix, as the integrals are replaced by some quadrature rule; see for instance Oosterlee and Vazquez [94].

For the HJB solution in the algorithm, apart from the optimization a sequence of linear equations has to be solved. The main task in the HJBVI part is to solve a *linear complementarity problem*, the Karush-Kuhn-Tucker conditions of a constrained quadratic programming problem in $\mathbb{R}^I$. This quadratic programming problem is the finite-dimensional analogue of the constrained variational minimization problem that a variational inequality poses. For more details, see [108] or Glowinski [56].

One standard technique for solving the linear complementarity problem for $\tilde{A} \in \mathbb{R}^{I \times I}$, $x, b, \psi \in \mathbb{R}^I$

$$\tilde{A}x - b \geq 0, \quad x - \psi \geq 0, \quad (\tilde{A}x - b) \cdot (x - \psi) = 0 \tag{3.40}$$

is the PSOR (projected successive over-relaxation) algorithm. As its name indicates, PSOR is derived from SOR, the iterative method to solve linear equations. The only difference to SOR for $\tilde{A}x = b$ is that after each iteration, the solution is projected back to the admissible cone $x \geq \psi$. For more on SOR and PSOR, see Hackbusch [59], [108], and Cryer [34].

We can now see clearly why in the optimal stopping iteration, the obstacle $\max(v^{n-1}, \mathcal{M}v^{n-1})$ has advantages over $\mathcal{M}v^{n-1}$: A good starting value for the PSOR is $x = \psi = \max(v^{n-1}, \mathcal{M}v^{n-1})$, especially îf v^n is close to v^{n-1} in an advanced iteration. Then the PSOR for the variational inequality will typically converge in a few iterations.

Remark 3.5.1. In the parabolic case presented here, the backward solution method for QVIs as described in §3.2 could be used alternatively. This direct method does not involve solving a linear complementarity problem, and is thus easier to implement. The main reason why we chose to present the above algorithm is that it is straightforward to deduce the elliptic analogue for it (at least without stochastic control), while the backward solution method is only applicable to parabolic problems.

3.5.2 Convergence analysis

In the convergence analysis for the discrete numerical approximation, two questions are of particular interest. First, does the value function of the discretized problem converge to the value function of the original problem for grid spacing $h \to 0$? Second, how optimal is the (discrete) optimal strategy derived from the discrete approximation, if we apply it to the original problem?

Convergence of discrete approximation

Let us denote by v_h^n the value function of the HJBVI (3.37) discretized with step $h > 0$. Assume that for $n \geq 0$ fixed, v_h^n converges to v^n for $h \to 0$, locally uniformly. Then under the assumptions of the convergence result of iterated optimal stopping (Theorem 3.4.6), there is a sequence $h = h(n) > 0$ such that $v_{h(n)}^n \to v$ locally uniformly for $n \to \infty$.

It thus remains to investigate the convergence of numerical methods for (HJB) variational inequalities (and for HJB equations in the case $n = 0$). There is a number of references for results of this kind, e.g., Jakobsen [67], Glowinski [56], and Fleming and Soner [46], Kushner and Dupuis [73] for HJB equations.

Let us roughly summarize how a proof of convergence can work. A fundamental insight formulated by Barles and Souganidis [14] in a viscosity solution framework is that a numerical scheme converges (to the right solution), if it is *monotone*, *stable*, and *consistent*, provided a comparison principle holds for the limit equation. Monotone means roughly that a weak ellipticity condition is satisfied, stable that the numerical solution u_h is bounded uniformly in h, and consistent that the true solution satisfies for $h \to 0$ the discretized equation.

Consider the HJBVI (3.13) with continuous obstacle g, take all assumptions for granted that are needed for viscosity solution existence and uniqueness. In particular, let comparison hold (Th. 3.3.4). We investigate a solution u_h of the discretized backward-time HJBVI with implicit timestepping

$$\begin{aligned} \min(-\sup_{\beta\in B}\{D_t^{1,h+}u + \mathcal{L}_h^\beta u + f^\beta\}, u-g) = 0 \qquad & \text{in } S_T \\ u - g = 0 \qquad & \text{in } \partial^+ S_T, \end{aligned} \tag{3.41}$$

where $D_t^{1,h+}u(t,x) = (u(t+h,x) - u(t,x))/h$ denotes the forward difference operator with timestep h. We recall from Chapter 1 the definitions $\overline{u}(x) = \limsup_{h\to 0, y\to x} u_h(y)$ and $\underline{u}(x) = \liminf_{h\to 0, y\to x} u_h(y)$. If we are able to prove that $\overline{u}$ is subsolution and $\underline{u}$ supersolution of the non-discretized HJBVI (3.13), then by the comparison principle, $u = \overline{u} = \underline{u}$ is the unique solution of (3.13), and $u_h \to u$ for $h \to 0$, locally uniformly.

We only sketch the proof that $\overline{u}$ is subsolution for $d = 1$, assuming that u_h is uniformly bounded in $\mathcal{PB}_p([0,T]\times\mathbb{R})$: Let $\varphi \in \mathcal{PB} \cap C^{1,2}([0,T]\times\mathbb{R})$ and $(t_0,x_0) \in [0,T]\times\mathbb{R}$ a (wlog strict) global maximum point of $\overline{u} - \varphi$. Then there exist sequences $h_n \to 0$, $(t_n,x_n) \to (t_0,x_0)$ with $u_{h_n}(t_n,x_n) \to \overline{u}(t_0,x_0)$ and such that $(t_n,x_n) \in [0,T]\times\mathbb{R}$ is a global maximum point of $u_{h_n} - \varphi$. Choose $\varphi_n := \varphi + u_{h_n}(t_n,x_n) - \varphi(t_n,x_n)$. Then for $(t_0,x_0) \in S_T$:

$$\begin{aligned} 0 &= \min(u_{h_n}(t_n,x_n) - u_{h_n}(t_n+h_n,x_n) - \sup_{\beta\in B}\{h_n\mathcal{L}_{h_n}^\beta u_{h_n}(t_n,x_n) + h_n f^\beta(t_n,x_n)\}, \\ &\qquad u_{h_n}(t_n,x_n) - g(t_n,x_n)) \\ &\geq \min(\varphi_n(t_n,x_n) - \varphi_n(t_n+h_n,x_n) - \sup_{\beta\in B}\{h_n\mathcal{L}_{h_n}^\beta \varphi_n(t_n,x_n) + h_n f^\beta(t_n,x_n)\}, \\ &\qquad u_{h_n}(t_n,x_n) - g(t_n,x_n)) \end{aligned} \tag{3.42}$$

if $\mathcal{L}_h^\beta u$ is monotone in nonlocal terms. This is the case if we set

$$\begin{aligned} \mathcal{L}_h^\beta u(t,x) = \frac{1}{2}\sigma^2 D_x^{2,h}u + \mu 1_{\mu>0} D_x^{1,h+}u + \mu 1_{\mu<0} D_x^{1,h-}u + \sum_{i\in\mathbb{Z}\setminus\{0\}} \big(u(t,x+ih) - u \\ - ih\,(D_x^{1,h-}u\,1_{i>0} + D_x^{1,h+}u\,1_{i<0})\,1_{ih\in\ell(\{|z|<1\})}\big)\ \nu(\{z : \ell(z) \in (ih - \frac{h}{2}, ih + \frac{h}{2}]\}), \end{aligned}$$

where $u = u(t,x)$, $\sigma = \sigma(t,x,\beta)$, $\mu = \mu(t,x,\beta)$, and $\ell(z) = \ell(t,x,\beta,z)$ was assumed to be strictly increasing in z. Here, the finite difference operators are

$$\begin{aligned} D_x^{2,h}u(t,x) &= \frac{u(t,x+h) - 2u(t,x) + u(t,x-h)}{h^2}, \\ D_x^{1,h+}u(t,x) &= \frac{u(t,x+h) - u(t,x)}{h}, \qquad D_x^{1,h-}u(t,x) = \frac{u(t,x) - u(t,x-h)}{h}. \end{aligned}$$

Now, taking limits $n \to \infty$ yields the convergence of (3.42) to

$$0 \geq \min(-\sup_{\beta\in B}\{\varphi_t + \mathcal{L}^\beta\varphi + f^\beta\}, \overline{u} - g),$$

where we have used the consistency of our finite difference approximations for $\varphi \in C^{1,2}$, the continuity of μ, σ, ℓ, the definition of $\mathcal{PB}$ and the compactness of the control set B. If the analogous relation is satisfied on $\partial^+ S_T$, then $\overline{u}$ is a subsolution of (3.13).

Remark 3.5.2. A monotonicity property of the finite difference approximation in the nonlocal terms as above (for fixed β) leads to a strictly diagonally dominant finite difference matrix with positive entries on the diagonal and nonpositive entries on the off-diagonals. This means that the M-matrix property is satisfied, which is known to be a desirable criterion in the numerical solution of a PDE (see, e.g., [102]).

Remark 3.5.3. The sketch of proof above reveals striking similarities to the proof of a stability theorem for (discontinuous) viscosity solutions (see §1.3, §3.3.4), where stability is meant also with respect to the partial differential operator $\mathcal{L}$. Indeed, all elements are the same apart from the monotonicity in the nonlocal terms, where in a stability proof the ellipticity condition is used instead. But note that we can interpret such a monotone scheme as a Markov chain: For example, the forward / backward difference operators depending on the sign of μ (also known as upwind/downwind discretization) correspond to the Markov chain jumping up with intensity μ (if $\mu > 0$), or jumping down with intensity $-\mu$ (if $\mu < 0$). The monotonicity condition of the numerical scheme is nothing else but the ellipticity condition needed for the interpretation as a Markov chain. While for a PDE, the nonlocal terms of a numerical scheme represent some generalization of a stability result, the convergence result for numerical schemes for a PIDE such as (3.13) is a mere consequence of a discontinuous stability result.

Convergence of the optimal strategy

We consider here again only the combined stochastic control and stopping problem, and denote its value function by $v(t,x) = \sup_{\beta\in\mathcal{B},\tau\in\mathcal{T}} J^{(\beta,\tau)}(t,x)$, where J is from (3.10). We define by

$$J_h^{(\beta,\tau)}(t,x) = \mathbb{E}^{(t,x)}\left[\int_t^\tau f(s, X_s^{\beta,h}, \beta_s)ds + g(\tau, X_\tau^{\beta,h})1_{\tau<\infty}\right], \tag{3.43}$$

the expected payoff of the discretized system, where $X^{\beta,h}$ is a Markov chain corresponding to the numerical discretization with grid spacing h. We allow not only discrete controls β and τ, but all $(\beta,\tau) \in \mathcal{B}\times\mathcal{T}$.

Then the question we want to discuss is whether an arbitrary optimal discrete strategy (β^h,τ^h) such that $J_h^{(\beta^h,\tau^h)} = v_h$ satisfies $J^{(\beta^h,\tau^h)} \to v$ locally uniformly, if $h \to 0$. Similar results arise when proving the convergence of numerical schemes by weak convergence of probability measures (see Kushner and Dupuis [73]). This gives already an idea that the answer can be found using convergence results about stochastic processes.

Indeed, by the convergence result for optimal stopping, we have $J_h^{(\beta^h,\tau^h)} \to v$ locally uniformly. Because for any norm $\|\cdot\|$,

$$\|J^{(\beta^h,\tau^h)} - v\| \le \|J^{(\beta^h,\tau^h)} - J_h^{(\beta^h,\tau^h)}\| + \|J_h^{(\beta^h,\tau^h)} - v\|,$$

if is sufficient to prove that $J_h^{(\beta,\tau)} \to J^{(\beta,\tau)}$ locally uniformly for $h \to 0$, uniformly in $(\beta,\tau) \in \mathcal{B}\times\mathcal{T}$. This is precisely the case if the discretized process $X^{\beta,h}$ converges weakly in distribution to X^β, uniformly in $(\beta,\tau) \in \mathcal{B}\times\mathcal{T}$. For results on such a uniform convergence in stochastic control, see Gikhman and Skorokhod [54], Th. 3.17.

The same result could in principle also be obtained by purely analytical techniques, if one proves the convergence of numerical schemes for linear PDE, uniformly in $(\beta,\tau) \in \mathcal{B}\times\mathcal{T}$ (i.e., uniformly in the stochastic control and the non-stopping domain).

3.5.3 Implementation details

We discuss here details of the implementation, mainly about the calculation of $\mathcal{M}u$, and the inclusion of boundary values. Most of the remarks are based on the experience from our example implementation. We recall that v_h^n denotes the discrete approximation of the n-th optimal stopping iterate v^n.

First, a few general remarks are in order. As already noted in §3.4.2, iterated optimal stopping is a good method especially if fixed costs are high and volatility is low, and thus with a high probability only few impulses suffice. For low fixed costs, it typically converges slower.

Among the different methods to solve an optimal stopping (or linear complementarity) problem, an iterative method like PSOR has the advantage that the numerical error can be controlled. It is more difficult to control the error over all optimal stopping iteration steps in a direct method such as the penalty method. A robust yet efficient way to choose the SOR relaxation parameter ω is an adaptive method as proposed by Hackbusch [59].

In a typical problem, the solution of the optimal stopping problem (3.37) via PSOR is much faster than the impulse optimization step or calculation of $\mathcal{M}u$.

Impulse maximization. Let us recall the definition of the intervention operator $\mathcal{M}$,

$$\mathcal{M}u(t,x) = \sup\{u(t,\Gamma(t,x,\zeta)) + K(t,x,\zeta) : \zeta \in Z(t,x)\}.$$

Small changes in v^{n-1} have no large effect on $\mathcal{M}v^{n-1}$, as $\mathcal{M}$ is in general a continuous operator by Lemma 2.4.3 (vi). However, the optimal impulses $\hat{\zeta}$ are very sensitive to changes in v^{n-1}, because the values on $\Gamma(t,x,Z(t,x))$ can be very similar.

From the discussion of the optimal strategy in §3.4.4, we know that in the approximation v^n, impulses are optimal in points (t,x) with $v^n(t,x) = \mathcal{M}v^{n-1}(t,x)$. In our numerical implementation, we label as *intervention region* all points where this equality holds to within a certain (very small) tolerance, and the optimal impulses are maximizers in $\mathcal{M}v_h^{n-1}(t,x)$.

Some further remarks:

- If we are on the discrete grid $(t_j,x_i)_{i,j}$, then some kind of interpolation is necessary for the impulse maximization, because typically not all points in $\Gamma(t,x,Z(t,x))$ will lie on the grid.

- In our test implementation, we first tried to use MATLAB's fmincon with a user-defined starting value to keep the solver as general as possible. However, fmincon sometimes finds a local, non-global minimum, especially if the starting values are not good. After these first negative experiences, we built our own optimizaton routine, which is inspired by genetic optimization algorithms. In the examples computed for one-dimensional impulse maximization, our method was very reliable in finding all global maxima, and almost as fast as fmincon.

- If $Z(t,x)$ is a bounded interval (and also for nonlinear transaction costs), then multi-step effects can occur: In the first optimization step, only one jump can be applied. After the next stopping iteration, it can be optimal to jump to a point where the value function already includes a jump. Especially if the costs are (affine-)linear, there can be an infinity of optimal impulses for each fixed starting point. Spikes in the computed optimal impulse may thus be seen as a warning that the optimal strategy is not unique.

- In the intervention region with $v^n = \mathcal{M}v^{n-1}$, the stochastic control obtained in the optimization is meaningless, because points in the intervention region are immediately left by an impulse.

Boundary values. Nonzero Dirichlet boundary conditions are usually included in the finite difference algorithm by adding a vector to $A^{\hat{\beta}}(\theta u^{(j)} + (1-\theta)u^{(j-1)})$ containing nonzero entries in lines i with $x_i \in \partial S$, and manipulating the other matrix entries suitably.

If S is unbounded, then we have to cut off the domain at some point, and impose artificial boundary conditions for the resulting computational domain. Frequently, these boundary conditions are not known explicitly; however they can be estimated by enlarging the computational domain, and taking only the results from a smaller subdomain. (Dirichlet) boundary conditions are also needed for the computation of integral terms which reach outside the computational domain.

An important point about the effective boundary conditions for $n \geq 1$ is that they are not prescribed by g in general; rather they may be changed by impulses. Luckily, there is nothing to do to allow for this: We recall from the proof of Proposition 3.4.4, that on the nonlocal boundary $\partial^+ S_T$,

$$\max(v^{n-1}, \mathcal{M}v^{n-1}) \geq g,$$

and hence also $\max(v_h^{n-1}, \mathcal{M}v_h^{n-1}) \geq g$ for the numerical solution. Thus if we set the boundary condition in the finite difference vector to g, then instead of g, the obstacle $\max(v_h^{n-1}, \mathcal{M}v_h^{n-1})$ is assumed on $\partial^+ S_T$.

A few additional remarks:

- An analogous procedure can be adopted for the cutoff boundary, if an estimate for the boundary condition is already available (e.g., from an enlarged domain in the first PDE iteration). For the subsequent (stopping) iterations, we need to enlarge the domain if outside there are impulses (or a significant impact of impulses), *and* our domain has a connection to this outside. (This could be also the case if outside, impulses are optimal at a future time, for a time-dependent equation.)

- If an impulse ends on or near the boundary, then the computational domain should be enlarged (or its shape changed). Impulses to outside the computational domain are problematic, because there typically only the Dirichlet boundary condition g is available. It is however possible to enlarge the computational domain by subsets of the nonlocal boundary $\partial^+ S_T$; on this part of the computational domain, simply the obstacle has to be taken as updated boundary condition (no linear complementarity problem).

- If the diffusion term and the drift term to the outside are 0 (and there are no infinite-activity jumps), then boundary conditions are not necessary — this corresponds to the real behaviour of the stochastic process. This means that the value on the boundary is determined by the rest of the domain, for parabolic problems in particular by the terminal condition.

Chapter 4

A dynamic model of credit securitization

As an application of the theory developed in the previous chapters, we present in this chapter a new model of bank credit risk management with combined impulse and stochastic control.

The plan of the chapter is as follows: First, we give an introduction to the topic and survey related literature. The first main section §4.2 introduces the model in detail, and is followed by a section culminating in the proof of the HJBQVI viscosity solution characteristics of the value function. After analyzing several stochastic control simplifications of the model in §4.4, we present and discuss numerical results in §4.5.

The contents of this chapter correspond in large parts to the working paper Frey and Seydel [50], which will be submitted to "Mathematics and Financial Economics".

4.1 Introduction

Banks staggered, stock prices plunged, governments had to intervene — the credit crisis starting in 2007 drew the public attention to a specific form of financial derivatives with loans as underlying that had been used to an enormous extent by banks all over the world. Complex credit securitization products such as Asset-Backed Securities (ABS) became known to a wider public as investments spreading American subprime home loans all over the world. Notwithstanding this negative connotation, credit securitization has its undeniable benefits: On the macro level, it can help to mitigate concentration risks within the banking sector; on the micro or firm-specific level, securitization is an important risk management tool as it enables an individual bank to reduce its leverage.

In this work, we are interested in securitization on the micro level and study the optimal dynamic securitization strategy of a commercial bank which is mainly engaged in lending activities. Transaction costs are an important factor in a securitization decision. We therefore incorporate fixed transaction costs (e.g., rating fees), and variable transaction costs (e.g., price discounts) into our model. In view of the fixed part of the transaction costs, it is natural to formulate and study the optimization problem in an impulse control setting.

The model. We consider a bank whose sole business is lending. For simplicity, the bank does not have customer deposits, and therefore refinances itself by debt capital. We assume that this refinancing is short-term, e.g., on the interbank market. The loans issued by the bank are modelled as a discrete portfolio of perpetuities which generate returns proportional to their nominal but may also default. These loans are valued on the bank's balance sheet at their nominal value, minus losses incurred (impairment). If the nominal value of the loans falls below the debt level, then the bank itself defaults. This risk of bank default however implies that the bank's refinancing rate may be higher than the risk-free interest rate.
In reality, loan default probabilities are uncertain and may change with the state of the economy. This leads us to consider a random state of the economy, modelled as a two-state continuous-time Markov chain. Correspondingly, also the market value of the loans and the bank's refinancing cost may change with the economic state.

We study the problem of maximizing the expected utility of the bank's liquidation value at some horizon $T > 0$. For this, the bank has two instruments at its disposal: on the one hand, it can sell loans at market value minus fixed transaction costs; this is modelled as securitization impulse. On the other hand, it can issue new loans; the decision whether to issue new loans is modelled as a standard stochastic control problem. Hence we have to deal with a so-called combined impulse and stochastic control problem. The analysis of this problem is the main technical contribution of this chapter.

Our model combines the most important factors affecting a bank's securitization decisions in a dynamic setting: loans may default and thus reduce profitability, or even jeopardize the existence of the bank; a securitization of loans can reduce risks, but the full nominal will probably not be recovered because of fixed and variable transaction costs depending on the current state of the economy; securitization can also be an alternative to refinancing via debt capital, especially if the latter is very expensive due to high refinancing costs. The fixed transaction costs in our model lead to finitely many securitization impulses. This mirrors the relative illiquidity of securitization markets and is in stark contrast to standard continuous-time portfolio optimization models with their assumption of continuous and costless portfolio rebalancing.

In our analysis, we carve out major challenges a bank faces in managing its loan exposure. Despite the complexity of the model, we are able to derive some theoretical results, and compute optimal solutions numerically. These results can serve as guidance for an optimal risk management strategy of a bank which is simultaneously active on the debt market, the securitization market and the retail market.

PDE approach. The value function of a combined impulse and stochastic control problem is known to be associated with a certain nonlinear, nonlocal partial differential equation (PDE), called the Hamilton-Jacobi-Bellman quasi-variational inequality (HJBQVI). The reader is probably already familiar with this type of equation from Chapter 2; for a different introduction to the subject, one may also consult Øksendal and Sulem [93], or Bensoussan and Lions [17]. Because we are dealing with a three-dimensional impulse control problem until terminal time, we cannot expect to find an analytical solution of the HJBQVI. This is also why standard verification techniques for smooth solutions fail in our case. So we have to consider weak solution concepts, such as viscosity solutions (see Chapter 1, or Crandall et al. [33], Fleming and Soner [46]), and to solve the problem by numerical techniques.

In this chapter, we prove in a very efficient manner that the value function of our combined impulse and stochastic control problem is the unique viscosity solution of a suitable HJBQVI, using our results from Chapter 2. Then, we can proceed to the numerical solution of this HJBQVI (which is done by iterated optimal stopping in a finite-difference scheme), and compute optimal impulse strategies for our problem.

Numerical results. The overall result from our analysis and numerical computations is that securitization is a valuable tool for a bank's credit risk management, especially if the initial leverage of the bank is high. The higher the bank's refinancing cost, the stronger is this incentive to securitize; this is in line with the general observation in the corporate finance literature that increasing costs to raising new external funds are an important rationale for risk management, see for instance Froot and Stein [51].

Our numerical results also demonstrate that transaction costs (fixed and variable) have a crucial impact in our model: First, different fixed transaction costs can lead to significant changes in the optimal impulse strategy. Second, there is a tendency to perform impulses when (proportional) transaction costs are lower. For our chosen set of parameters, this means that impulses in expansion (where the market value of loans is higher and hence transaction costs lower) are optimal in a relatively large region although loans are profitable in such boom times; such impulses in expansion serve as a provision for bad times.

Under the plausible assumption of a strongly procyclical market value of loans, impulses near the default boundary of the bank are simply not admissible in recession, because this would lead to immediate default. The optimal (impulse) strategy in this case is simply to wait for better times. This effect can be observed — although less pronounced — also for a weakly procyclical market value of loans. *If* the bank decides to do a securitization in recession, then it should only securitize a relatively small amount due to the proportional transaction costs. The irony is that when the model bank desperately needs capital relief and leverage reduction, it is only available at (prohibitively) high cost.

Literature. The problem of choosing the optimal leverage for a firm is a classical problem in corporate finance, see for instance Leland and Toft [75], Ziegler [112], or Hackbarth et al. [58]; the problem is analyzed specifically for banks in Froot and Stein [51], and an empirical analysis for a commercial bank is carried out in Cebenoyan and Strahan [25]. Here, we concentrate not on this theoretical question, but investigate the problem for a bank from a transaction-based perspective, i.e., "what should the bank optimally do, if in a certain (non-optimal) situation?". Another area of research related to our problem is optimal control for insurers (see, e.g., Schmidli [105]), in particular optimal reinsurance (e.g., Irgens and Paulsen [63] and references therein). Some further background on ABS, securitization and credit risk management can be found in Benvegnu et al. [20], Bluhm et al. [22], Franke and Krahnen [49] and McNeil et al. [81].

The novel features of our control problem (as opposed to standard continuous-time portfolio optimization problems such as Merton [84], [85]) are the inclusion of jumps and the use of impulse control methods. Portfolio optimization with jumps has been studied in Framstad et al. [48], among others; impulse control techniques have previously been used by, e.g., Eastham and Hastings [40] or Korn [71]. Some further references are given in Øksendal and Sulem [93].

Overview of the chapter. The first main section §4.2 introduces the model in detail, and discusses several aspects of choosing functional forms for market value und transaction costs. In the following §4.3, the linear boundedness of the value function is shown, and we establish that the value function is the viscosity solution of the HJBQVI. After analyzing several stochastic control simplifications of the model in §4.4, we describe in §4.5 the numerical algorithm used for the solution of the HJBQVI, and present and discuss numerical results.

4.2 The model

4.2.1 Basic structure

The bank. We consider a bank whose only business is lending, and we assume that the bank does not have customer deposits. In this simplified setting, the balance sheet consists only of equity, debt capital, cash and loans. These four factors then determine success or failure of the bank. Our model is based on the fundamental balance sheet equation

$$\begin{aligned} \text{assets} &= \text{liabilities} \\ \text{cash} + \text{loans} &= \text{equity} + \text{debt capital.} \end{aligned}$$

A bank of this type would normally refinance the issued loans to a large proportion by debt capital (a typical bank actually owns less than 10% of the assets on its balance sheet). As the structure of long-term debt capital typically remains largely unchanged over a longer time horizon, we make here the assumption that the long-term debt capital is constant and — for simplicity — equal to 0.

In this simplified setting, the bank refinances itself through a negative cash position, which is interpreted as short-term refinancing, say on the LIBOR interbank market.[1] We stress that negative cash in our model does *not* lead to immediate bankruptcy, but is just an indication that the bank does not own all of the assets on its balance sheet. Relying on short-term funding has been quite a common way for banks to refinance itself, at least until the fall of 2008. Indeed, lending long-term and refinancing short-term is one the *raisons d'être* of banks. The drawback of short-term refinancing however is that the refinancing rate can be quite sensitive to changes in the bank's situation or in the economic environment, or that the bank might even not be able to raise funds at all. This became evident in September and October 2008, when, e.g., the German bank Hypo Real Estate, and several US investment banks were on the brink of bankruptcy and could only be saved by government intervention.

In the balance sheet equation described above, three factors remain, of which we choose to model the nominal value of loans L and cash C, and deduce equity $E = L + C$. As banks cannot take a short position in loans, $L \geq 0$, whereas the sign of C is not restricted. The bank exists as long as $E \geq 0$, otherwise default occurs. We define the *leverage* of a bank as follows:

$$\text{leverage} = \frac{L}{E} = \frac{L}{L+C} \in [0, \infty].$$

A leverage > 1 means that $C < 0$ (refinancing of some of the loans on the short-term debt market) and reversely, a leverage $\in [0, 1]$ means that $C \geq 0$, so that the bank owns all its assets. A high leverage indicates a high riskiness of the bank, should loans default. In this case, we would expect a higher refinancing rate for the bank.

The dynamic model. We now present our three-dimensional model (first without securitization) step by step. Denote by $X = (L, C, M)^T$ the stochastic process composed of loan value

[1] Strictly speaking, this means that cash could be on either side of the balance sheet, depending on whether it is positive or negative.

L, cash C, and state of the economy M. Our model lives on a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $(\mathcal{F}_t)_{t\geq 0}$, satisfying the usual assumptions.

The **loan portfolio** of the bank is discrete, i.e., at every instant, it consists of finitely many loans. Furthermore, the portfolio is homogeneous, i.e., all loans have the same interest rate, the same risk and the same nominal; without loss of generality we assume for each loan a nominal of 1. Each loan has maturity ∞ (perpetuity), and defaults with a certain intensity (independent from the other loans conditionally on the state of the economy), upon which it is immediately liquidated. The nominal value of the loan exposure L develops in time according to an adapted càdlàg point process with varying intensity:

$$dL_t = -dN_t + \beta_t dP_t, \qquad L_0 \in \mathbb{N}_0. \tag{4.1.L}$$

Here, N_t is a Poisson process with intensity $\lambda(M_{t-})L_{t-}$ dependent on the state of the economy M (see below) and the current loan nominal L. This process can be derived from the individual defaults of the loans as follows: Loans default with intensity $\lambda(M_{t-})$, independent conditionally on M. For a total portfolio of L_{t-} loans, the intensity of one loan defaulting is thus $\lambda(M_{t-})L_{t-}$. P is an adapted càdlàg standard Poisson process (independent of N) with intensity $\lambda_P \geq 0$, and β is a predictable stochastic control process with values in $\{0, 1\}$. This control gives the possibility to increase the loan nominal, should there be an opportunity: a value $\beta = 1$ means green traffic light if a customer comes into the bank and asks for a loan. Note that in this way, we ensure that $L_t \in \mathbb{Z}$ for all $t \geq 0$, i.e., that the loan portfolio stay discrete.

The **cash process** C evolves according to the following SDE (recall $X = (L, C, M)^T$):

$$dC_t = (r_B(X_t)C_t + r_L L_t)\, dt + (1 - \delta(M_{t-}))dN_t - \beta_t dP_t. \tag{4.1.C}$$

Here, the measurable function $r_B \geq 0$ is the instantaneous refinancing rate of the bank (or interest rate earned on cash if $C_t > 0$), and will depend on the riskiness of the bank, in particular on its leverage. In modelling refinancing by an instantaneous cash flow stream instead of the usual three- or six-month horizon on the LIBOR market, we ensure the Markov property of X and thus numerical tractability; in §4.2.2 we will present examples how to choose the refinancing function. In general, $r_B \geq \rho$ for the risk-free interest rate $\rho \geq 0$. Note that the existence of such a function r_B implies that we assume there is always refinancing available, regardless how risky the bank is. The constant r_L is the continuous rate all customers have to pay for their loans. The remaining terms on the right hand side of (4.1.C) are already known from the discussion of the loan process: $\delta(M_{t-}) \in [0, 1]$ represents the current loss given default (LGD), so that the term $1 - \delta$ is the recovery rate from the liquidation of a defaulted loan; $\beta_t dP_t$ represents the money that is invested for issuing new loans.

Finally, the **economy process** M is an adapted càdlàg Markov switching process or continuous-time Markov chain with values in $\{0, 1\}$ (expansion, contraction) and switching intensities $\lambda_{01}, \lambda_{10} > 0$, with λ_{01} being the intensity to go from 0 to 1. M is assumed to be independent of all other processes encountered so far. Formally, M can be represented as difference of two independent Poisson processes N^{01} and N^{10}:

$$dM_t = 1_{\{M_{t-}=0\}} dN_t^{01} - 1_{\{M_{t-}=1\}} dN_t^{10}. \tag{4.1.M}$$

We consider here only the simple case of two states of the economy; more states (or even a more complex economy process, as long as its stays Markov) can be handled in the same way.

The bank's interventions. The bank wants to maximize its expected terminal utility by controlling its loan exposure, which might be too high and thus too risky, or too low to generate significant profits. This maximization can be done either by issuing new loans (control of β), or it can be done via securitization. Securitization is a means to get loans off the balance sheet, but a securitization comes always with certain fixed costs $c_f > 0$, such as rating agency fees, or legal costs for setting up a special purpose vehicle in a tax haven. Moreover, there may be variable transaction costs, as the securitizing bank may not be able to sell the loans for the value attributed to them on its balance sheet.

Securitization is modelled as impulse control because of the transaction costs. A securitization impulse reduces the loan exposure by ζ, and the cash is increased by the market value $\eta(x_3, \zeta)$ minus fixed costs c_f. We assume that $\eta(x_3, 0) = 0$, that $\eta \geq 0$, and that η is monotonically increasing in the second component. In mathematical terms, the effect of a securitization impulse of ζ loans is to bring the process X from the state $x \in \mathbb{N}_0 \times \mathbb{R} \times \{0, 1\}$ to the new state

$$\Gamma(x, \zeta) = (x_1 - \zeta, x_2 + \eta(x_3, \zeta) - c_f, x_3)^T, \tag{4.2}$$

where ^{T} denotes the transpose. The distinction between the nominal value L in (4.1.L) used in accounting and the market value that investors are willing to pay will be particularly important for our model. A possible choice for the market value function η will be presented in §4.2.2.

Let us denote the impulse control strategy by $\gamma = (\tau_1, \tau_2, \ldots, \zeta_1, \zeta_2, \ldots)$, where τ_i are stopping times with $0 = \tau_0 \leq \tau_1 \leq \tau_2 \leq \ldots$, and ζ_i are $\mathcal{F}_{\tau_i}$-measurable impulses. We admit only impulses ζ_i that are in the set $\{0, \ldots, L_{\tau_i -}\}$. By $\alpha = (\beta, \gamma) \in \mathcal{A} = \mathcal{A}(t, x)$, we denote the so-called combined stochastic control, and $\mathcal{A}(t, x)$ denotes the set of admissible combined stochastic controls. $\mathcal{A}(t, x)$ is chosen such that existence and uniqueness of the SDEs (4.1.*) holds for all admissible controls. To ensure that the controlled process is Markov, we additionally require that $\alpha \in \mathcal{A}$ be Markov in the sense that τ_i are first exit times of $(t, X_t)_{t \geq 0}$, $\zeta_i \in \sigma(\tau_i, X_{\tau_i})$ and $\beta_t \in \sigma(t, X_{t-})$. In the next section it will be shown that $\mathcal{A}$ is non-empty.

The controlled process $X^\alpha = (L^\alpha, C^\alpha, M)$ is determined by the SDEs (4.1.L), (4.1.C) and (4.1.M) between the impulses, and at τ_{i+1} changed by the impulses:

$$X_{\tau_{i+1}} = \Gamma(\check{X}_{\tau_{i+1} -}, \zeta_{i+1}) \qquad i \in \mathbb{N}_0, \tag{4.1.I}$$

where the term $\check{X}^\alpha_{\tau_j -}$ denotes the value of the controlled process X^α in τ_j including a possible jump of the process, but excluding the impulse, i.e., $\check{X}^\alpha_{\tau_j -} = X^\alpha_{\tau_j -} + \Delta X^\alpha_{\tau_j}$.

The optimization problem. We consider the optimization problem of the bank on the domain

$$S = \{x : x_1 > -1,\ x_1 + x_2 > 0\} \subset \mathscr{S} := \mathbb{N}_0 \times \mathbb{R} \times \{0, 1\},$$

i.e., as long as the bank does not default, and as the nominal value of the loans is nonnegative. The stopping time $\tau_S = \inf\{s \geq t : X^\alpha_s \notin S\}$ denotes the first exit time from S. Note that exit from S can only occur on $\{x_1 + x_2 = 0\}$, so shorting loans is not possible. We allow the case $L = 0$ for $C > 0$ although the bank in this case suspends its business; it may continue its business later on by setting $\beta = 1$, i.e., by issuing new loans.

The objective of the bank is to find a strategy $\alpha = (\beta, \gamma) \in \mathcal{A}$ that maximizes the expected utility of its liquidation value at some horizon date T. Consider a utility function $U : \mathbb{R}^+_0 \to \mathbb{R} \cup \{-\infty\}$ and assume that U is strictly increasing and concave on $[0, \infty)$. Define for $\alpha \in \mathcal{A}$, $t \leq T$ and $x \in \mathscr{S}$ the function

$$J^{(\alpha)}(t, x) = \mathbb{E}^{(t,x)} \left[U \left(\max\{\eta(M_\tau, L^\alpha_\tau) + C^\alpha_\tau, 0\} \right) \right], \quad \text{with } \tau := \tau_S \wedge T. \tag{4.3}$$

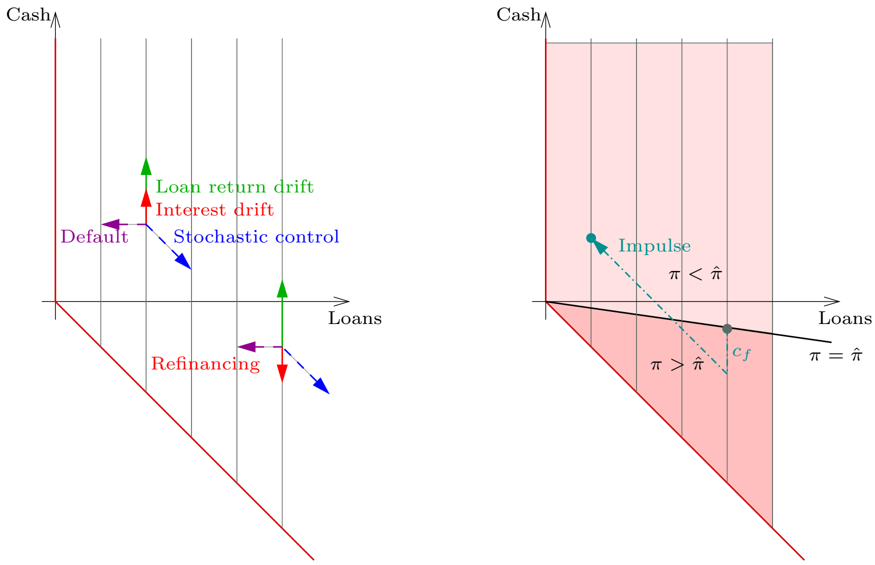

Figure 4.1: Visualization of the SDE terms for $\delta = 1$ (at left) and impulse graph for $\eta(x_3, \zeta) = \zeta$ (at right). Both are depicted in a (L, C) graph for fixed economy. The shaded regions in the right graph indicate whether the leverage π is greater, equal or smaller to the leverage $\hat{\pi}$ at the point of departure

Then the value function v of the bank's optimization problem is defined by

$$v(t, x) = \sup_{\alpha \in \mathcal{A}(t,x)} J^{(\alpha)}(t, x). \tag{4.4}$$

For future use we define $g(x) := U(\max\{\eta(x_3, x_1) + x_2, 0\})$, such that $J^{(\alpha)}(t, x) = \mathbb{E}^{(t,x)}[g(X_\tau^\alpha)]$.

Remark 4.2.1. As economic interpretation, the objective function in the optimization problem (4.3), $g(X_\tau^\alpha)$, can be viewed as utility of a majority shareholder, when the bank is liquidated at the horizon date T.

Remark 4.2.2. "Endogenous bankrupcty" as used in Leland and Toft [75], i.e., the possibility of the shareholders to liquidate the firm at any time, is automatically included in our setting: An impulse to $L = 0$, and then deciding to stay there by $\beta = 0$, terminates the business of the bank; yet still the interest ρ accumulates until T.

4.2.2 Refinancing rate r_B and market value η

In this subsection, we discuss building principles and examples for the refinancing rate r_B and the market value η. While it is relatively easy to find a good form for η, the discussion on r_B is considerably more involved. Let us emphasize that the functions proposed here are just *ad hoc* choices: they can be motivated from the model, but can not be strictly derived from it, so are not model-endogenous.

Market value. The market value is the amount for which loans can be sold on the secondary market. For our definition of the market value, the starting point is what we call the "fundamental" or risk-neutral value of one loan. Formally, this quantity is given by

$$p_m^\infty := \mathbb{E}\left[\int_0^\tau e^{-\rho s} r_L ds + e^{-\rho\tau}(1-\delta(M_\tau)) \middle| M_0 = m \right],$$

where τ stands for the default time of the loan. We recall that r_L is the loan interest rate, ρ the risk-free interest rate, and the functions λ, δ represent the relative loan default intensity and loss given loan default, respectively. In the special case where λ and δ are constant, p_m^∞ is independent of m and given by $\frac{r_L+(1-\delta)\lambda}{\rho+\lambda}$. In the general case, p_m^∞ can be obtained by a simple matrix inversion (see §A.3).

We assume that investors in securitization markets are risk-averse, so that the market value will typically be lower than the fundamental value.

Example 4.2.1. The following form for η is used in our numerical examples (§4.5): We apply to the risk-neutral value a procyclical factor to reflect risk aversion, and cap the resulting value at ζ (i.e., the bank can not obtain more than the nominal value to avoid obvious arbitrage):

$$\begin{aligned} (a) \qquad & \eta_a(m,\zeta) := \zeta \cdot \min\left(1,\ p_m^\infty \cdot (1-(m+1)\delta(m)\lambda(m))\ \right) \\ (b) \qquad & \eta_b(m,\zeta) := \zeta \cdot \min\left(1,\ p_m^\infty \cdot (1-\delta(m)\lambda(m))\ \right) \end{aligned}$$

Recall that $m = 0$ in expansion, so the only effect of the factor $(m+1)$ is to double the default intensity in contraction. The procyclical factor can be interpreted as a form of overcollateralization of the ABS, i.e., the bank has to put more loans into the pool such that the expected first-/second-year losses are covered without affecting the investors.

Refinancing rate. A constant refinancing rate of the bank r_B would mean that the bank could raise money at a rate independent of its leverage and the riskiness of its loan portfolio. As this is certainly an unrealistic assumption, we have to think about a functional form of r_B incorporating the main risk factors of the bank in our model. Every reasonable choice for r_B should certainly be monotonically increasing in loan default rates, and also in the leverage of the bank. Furthermore, for $C > 0$ and hence leverage < 1, r_B should be equal to the risk-free rate ρ, as there is no risk of bank default.

To ensure these properties, we use as point of departure the following basic rule of thumb: On average, the bank's creditor wants to earn the annualized risk-free interest ρ. Given a lending horizon h, he will demand a refinancing rate r_B according to

$$1 + h\rho = PD \cdot (1 - LGD) + (1 - PD) \cdot (1 + hr_B). \tag{4.5}$$

Here $PD = PD(h)$ and $LGD \in [0, 1]$ represent creditors' perception of the default probability of the bank over the horizon h and of its loss given default, respectively.[2]

All quantities in (4.5) except ρ can be dependent on the current state $(\ell, c, m) \in S$ — in the following, we will mostly omit this argument for ease of notation. Equation (4.5) leads to the definition for $r_B = r_B(\ell, c, m)$

$$r_B := \frac{h\rho + PD \cdot LGD}{h(1 - PD)}. \tag{4.6}$$

Note that as required, for $PD = 0$, we have $r_B = \rho$. Now the only quantity left to model is the PD. Without loss of generality, consider the case $t = 0$. First, for a given loan amount ℓ and

[2]Of course this perception of the default probability of the bank could also (and in practice will) depend on r_B. For simplicity, we chose to ignore this secondary effect in our model.

cash position c, the PD can be defined as the probability that loan losses exceed current equity capital $\ell + c$:

$$PD := \mathbb{P}(-\Delta L > \ell + c) = \mathbb{P}\left(\frac{-\Delta L}{\ell} > \frac{\ell + c}{\ell}\right) \tag{4.7}$$

Hence we need to model the distribution of the $[0, 1]$-valued relative loss $-\Delta L/\ell$ over horizon h.

It would be natural to model the relative loss using a (discrete) Bernoulli mixture distribution for the following reason: Given the trajectory of M, the loan defaults at a given horizon date t are identically independent Bernoulli distributed, so that L_t in our model follows a Bernoulli mixture model with mixing over the different economy states (cf. McNeil et al. [81], Bluhm et al. [22]). However, it is easier to specify a continuous distribution which does not depend on the granularity of the portfolio; one can further argue that a continuous, or even smooth function r_B is reasonable because in reality the bank's creditor does not have full information about the bank's parameters and current state.

Example 4.2.2. For our numerical examples in §4.5, we take recourse to the Vasicek portfolio loss distribution. The Vasicek loss distribution arises as limiting case of a probit-normal Bernoulli mixture distribution for an infinitely granular portfolio, that is for $\ell \to \infty$; see Bluhm et al. [22] or the more general Prop. 8.15 in McNeil et al. [81]. Its distribution function is $V_{p,\varrho}(x) = N\left[1/\sqrt{\varrho}(N^{-1}(x)\sqrt{1-\varrho} - N^{-1}(p))\right]$, where N (N^{-1}) is the cumulative (inverse) normal distribution function. The parameter $p \in (0, 1)$ has the interpretation of an average default rate, $\varrho \in (0, 1)$ is a correlation parameter that models how much the default rate of a single loan varies with a common factor, such as the economic state M. With this choice the default probability of the bank is given by

$$(a) \quad PD := 1 - V_{p,\varrho}\left(\frac{\ell + c}{\ell}\right).$$

$p = p(m)$ will normally be chosen close to the current default intensity in our model, reflecting the short-term horizon of the refinancing. The parameter $\varrho = \varrho(m)$ can be used to model risk aversion on the part of the bank's creditors, arising for instance from incomplete information regarding the current state of the bank. We will use in our numerical examples also another form, which takes into account the proceeds from the loans (assuming a refinancing rate of ρ to avoid a circular reference):

$$(b) \quad PD := 1 - V_{p,\varrho}\left(\frac{(1 + r_L)\ell + (1 + \rho)c}{(1 + r_L)\ell}\right)$$

For the first form, if $\ell + c = 0$, then PD will be 1 and thus $r_B = \infty$. The second form leads for $r_L > \rho$ always to a $PD < 1$ and thus to finite r_B.

Remark 4.2.3. The continuous probit-normal mixing distribution underlying the Vasicek distribution in Example 4.2.2 would correspond to infinitely many economic states in our model. Notably, the effective default intensity implied by the Vasicek distribution is unbounded; in contrast, the mixing distribution in our model with two economic states only assumes values in two states determined by the two possible default intensities.

We stress that already via the mere *existence* of a refinancing function r_B, we assume that there is always refinancing available. If refinancing were not available (e.g., because PD is dependent on r_B, and there is no solution to (4.5)), then default would occur not at the boundary ∂S, but already inside S. This would then give rise to an endogenous default definition via backward induction, and thus further complicate matters.

4.3 Properties of the value function

This section collects a few technical properties of the model and the value function.

First of all, we note that existence and uniqueness of the SDE defined in (4.1.L), (4.1.C), (4.1.M) for constant β follows from Theorem V.3.7 in Protter [101], provided that (process) Lipschitz conditions on $r_B C$ are satisfied: A Poisson process with state-dependent intensity (without explosion time) is a semimartingale, so L is a well-defined semimartingale, too; the same holds for M. The process C is well-defined on S if

$$\sup_{c \geq -L_{t-}+\varepsilon} |r_B(L_{t-}, c, M_{t-})| \tag{4.8}$$

is an adapted càglàd process for each $\varepsilon > 0$, which is true as long as the sup in (4.8) exists for constant L, M (because L and M are step processes). In particular, the condition is satisfied for a constant r_B and the r_B examples given in §4.2.2, Example 4.2.2.
We can conclude that $\mathcal{A}(t,x)$ is non-empty.

The value function of a combined stochastic and impulse control problem is known to be associated with a certain partial integro-differential equation (PIDE), called the Hamilton-Jacobi-Bellman quasi-variational inequality (HJBQVI); see §2.2 for the general case, and §4.3.2 for the present case. Let $S_T := [0,T) \times S$, and define its parabolic "boundary" $\partial^+ S_T := ([0,T) \times S^c) \cup (\{T\} \times \mathscr{S})$, where the complement is taken in $\mathscr{S} = \mathbb{N}_0 \times \mathbb{R} \times \{0,1\}$. Then the HJBQVI in our setting takes the form

$$\begin{aligned} \min(-\sup_{\beta \in \{0,1\}} \{u_t + \mathcal{L}^\beta u\}, u - \mathcal{M}u) &= 0 \qquad \text{in } S_T \\ \min(u - g, u - \mathcal{M}u) &= 0 \qquad \text{in } \partial^+ S_T, \end{aligned} \tag{4.9}$$

where $\mathcal{L}^\beta$ is the infinitesimal generator of the state process X defined by the SDE (4.1.*): with $\tilde{x} := (x_1, x_2)$,

$$\begin{aligned} \mathcal{L}^\beta u(x) = \left(u(\tilde{x} + \begin{pmatrix} -1 \\ 1-\delta(x_3) \end{pmatrix}, x_3) - u(x) \right) \lambda(x_3) x_1 + \left(u(\tilde{x} + \begin{pmatrix} \beta \\ -\beta \end{pmatrix}, x_3) - u(x) \right) \lambda_P \\ + (u(\tilde{x}, 1-x_3) - u(x))\, \lambda_{x_3,(1-x_3)} + (r_B(x) x_2 + r_L x_1) u_{x_2}. \end{aligned}$$

Finally, the impulse intervention operator $\mathcal{M} = \mathcal{M}^{(t,x)}$ is defined to be

$$\mathcal{M}u(t,x) = \sup\{u(t, \Gamma(x,\zeta)) : \zeta \in \{0, \dots, x_1\}\}. \tag{4.10}$$

Intuitively, the condition $v - \mathcal{M}v \geq 0$ means that an impulse can not improve the value function v. The inequality $\sup_{\beta \in \{0,1\}}\{v_t + \mathcal{L}^\beta v\} \leq 0$ then suggests that under all possible strategies, $v(t, X_t^\alpha)$ is a supermartingale, so decreases in expectation. In any point $(t,x) \in S_T$, either $v = \mathcal{M}v$ has to hold (an impulse takes place), or $\sup_{\beta \in \{0,1\}}\{v_t + \mathcal{L}^\beta v\} = 0$, i.e., the stochastic process evolves according to the optimally controlled SDEs (4.1.L), (4.1.C), (4.1.M).

The PDE (4.9), corresponding to the full problem as exposed in §4.2, has no known analytical solution. First and foremost, this is because impulse control until a terminal time is very difficult, if not impossible to solve explicitly (because the smooth-fit property that works in the time-independent case has to be applied on a curve). In our case, the high dimensionality makes it very unlikely for such strategies to succeed, even in the time-independent or elliptic case.

4.3.1 Bounds for the value function

We want to prove that the value function is bounded (linearly) from below and above. While the first is immediate if U is bounded from below, the second necessitates some work.

In the following, we will use that if we admit general adapted controls, then this will not change our value function, i.e., it suffices to consider Markov controls. For proofs of this fact in stochastic control, we refer to Øksendal [89], Haussmann [61] or El Karoui et al. [42].

Proposition 4.3.1. *The function $c \mapsto v(t, \ell, c, m)$ is increasing for all $t \in [0, T]$, $(\ell, c, m) \in S$; it is strictly increasing if $r_B > 0$.*

Proof: For a given admissible combined control strategy α, we fix this strategy dependent on the events of $X^{\alpha,t,x}$ started in $X_t = x \in S$. Consider $X^{\alpha,t,y}$ for a y with all components equal to x, but $y_2 > x_2$ (more cash). As a concatenation of (strictly) increasing functions (SDE, impulses, and U), $g(X_\tau^{\alpha,t,y}) \geq (>)\ g(X_\tau^{\alpha,t,x})$. Note that α is in general not a Markov strategy of X started in y, but only adapted to $(\mathcal{F}_s)_{s \leq t}$ (this is why the optimality of Markov controls is needed as prerequisite). □

To be able to prove that the value function is bounded, we need an upper bound on the (optimal) leverage π. In the original setting of (4.1.*), this problem is elegantly resolved by the "business arrival process" P with its finite intensity. On the one hand, this means that the leverage can only be increased if P jumps. On the other hand, we will see that this implies the linear boundedness of the value function: The initial leverage is automatically reduced by loan proceeds (which accumulate in the cash account) — the finite intensity of P makes sure that there is a natural upper bound to shifting back these proceeds. In business terms, this may be interpreted as the potential demand of the customer base being finite.

We denote in the following by S^c the complement of the domain $S \subset \mathscr{S}$, and $S^+ := \{x \in S, x_2 > 0\}$, $S^- := \{x \in S, x_2 < 0\}$.

Proposition 4.3.2. *The value function v is linearly bounded from above if $\tilde{\rho} := \sup_{x \in S^+} r_B(x) < \infty$, $\tilde{\rho} \leq \sup_{x \in S^-} r_B(x)$ (roughly: refinancing cost greater than risk-free interest rate), and $\eta(\cdot, \zeta) \leq b\zeta$ for some $b > 0$.*

Proof: We bound the impulse control value function by the value function of a stochastic control problem on S; the upper bound for v on S^c then immediately follows.

Without loss of generality, we assume $r_L > \tilde{\rho}$. The original impulse control value function is (by Prop. 4.3.1) bounded by the value function of the problem without fixed or proportional transaction costs (i.e., $c_f = 0$ and $\eta(m, \zeta) \geq \zeta$), and $r_B \equiv \tilde{\rho}$ (which is typically the risk-free interest rate ρ). Without transaction costs, it is clear that we will obtain another upper bound if we place ourselves in expansion without loan defaults. A further upper estimate can be obtained if we allow impulses up to the amount of loans, but without deducting the securitized amount from L. The resulting optimally controlled process follows the SDE (starting wlog in $t = 0$):

$$\begin{aligned} dL_t &= dP_t, \qquad L_0 = \ell \\ dC_t &= (r_L L_t + \tilde{\rho} C_t)dt + b dP_t, \qquad C_0 = c + b\ell, \end{aligned} \tag{4.11}$$

where of course the optimal strategy was to have maximal leverage (by construction of (4.11), all possible impulse benefits are already included at no cost). Now we can assume that the P

jumps happen immediately in 0, and the value function can be bounded as follows (with the definition $\tilde{U}(x) := U(\max\{x, 0\})$):

$$\begin{aligned}
v(0,\ell,c) &= \mathbb{E}[\tilde{U}(\eta(0,L_T)+C_T)] \\
&\le \sum_{q=0}^{\infty} \mathbb{E}\left[\tilde{U}\left(\eta(0,\ell+q)+(c+b\ell+bq)\exp(\tilde{\rho}T)+(\ell+q)\frac{r_L}{\tilde{\rho}}(\exp(\tilde{\rho}T)-1)\right)\right]\mathbb{P}(P_T=q) \\
&\le \sum_{q=0}^{\infty} \tilde{U}\left(C_1\ell+C_2c+C_3q\right)\frac{(\lambda_P T)^q}{q!} \\
&\le C_1\ell+C_2c+C_3
\end{aligned}$$

by the increasingness and concavity of U, for generic constants C_i dependent on T. □

It is trivial that v is bounded from below if U is bounded from below. If U is not bounded from below (e.g., $U(0) = -\infty$ as in case of a log-utility function), then the existence of a lower bound is a question of controllability. We can make sure that $v(t,x) > -\infty$ if there is an $\alpha \in \mathcal{A}(t,x)$ and an $\varepsilon = \varepsilon(t,x) > 0$ such that $\mathbb{P}^{(t,x)}(L_\tau^\alpha + C_\tau^\alpha \le \varepsilon) = 0$. This is the case if there is an impulse control that immediately puts the bank permanently out of danger. Boundedness from below thus holds if these strategies exist with uniform $\varepsilon > 0$, which can only be the case if $\eta(x_3,\zeta) \ge \zeta$.

Remark 4.3.1. For the proof of Prop. 4.3.2, we could also have used a verification theorem in the style of Øksendal and Sulem [93], Theorem 8.1. We chose the above approach because the strict increasingness property is useful in itself and less abstract.

4.3.2 Viscosity solution property

In this subsection, we will prove that the value function of our combined stochastic and impulse control problem (4.4) is the unique viscosity solution of the HJBQVI (4.9). The proof consists mainly in checking that the assumptions of the general results in Theorem 2.2.2 are satisfied. See the references in Chapter 2, or the introductory Chapter 1 for more information on viscosity solutions in connection with stochastic control.

We recall the conditions in §2.2 for v to be a (unique) viscosity solution of (4.9) can be roughly summarized as follows: (Lipschitz) continuity of functions involved ((V*), (B*) and (E2) conditions), polynomial boundedness of the value function (E1), and continuity of v at the boundary (E3). Furthermore, we need the existence of a strict supersolution w.

We note that our setting here is slightly different from the setting in Chapter 2 in two main respects: (a) discrete state variables, and (b) state-dependent intensity. The proofs in Chapter 2 however adapt readily to (a) with effectively no continuity requirements in the discrete variables. We have already discussed in §2.6 how the results can be extended to problems with state-dependent intensity.

Theorem 4.3.3. *Assume that $c \mapsto r_B(\ell,c,m)$ is continuous $\forall(\ell,c,m) \in S$, and U be continuous and bounded from below. Further assume that $\liminf_{c\downarrow-\ell} r_B(\ell,c,\cdot) > r_L$ for $\ell > 0$, that $\tilde{\rho} := \sup_{x\in S^+} r_B(x) < \infty$, $\tilde{\rho} \le \sup_{x\in S^-} r_B(x)$ (roughly: refinancing cost greater than risk-free interest rate), and $\eta(\cdot,\zeta) \le \zeta$. Then the value function v in (4.4) is the unique viscosity solution of (4.9) in the class of linearly bounded functions, and it is continuous on $[0,T] \times \mathscr{S}$ (i.e., continuous in time and in cash).*

Proof: In general, continuity requirements have to hold only in (t, x_2) (time and cash), because loans and economy are discrete state variables.

(V1), (B1) hold because of discreteness, (V3), (E2) and (B2) by discreteness and assumption. (V2) is satisfied because the Hausdorff convergence in discrete loan dimension does not have to hold (non-emptyness holds wlog because for $x_1 < 0$, we can set the intervention set to $\{0\}$ without affecting the value function).
(U1), (U2) do not need to hold because the jump measures are finite. (V4) holds trivially again because of the finiteness of the jump measures; the set $\mathcal{PB}$ can be defined with an arbitrary polynomial. (E4) holds, e.g., by setting $\hat{\beta} := 10$. (E1) holds because of Proposition 4.3.2.

(E3) only needs to hold for $t_n \to t \in (0, T]$, $c_n \downarrow -\ell$ due to the loan discreteness. Wlog, control / interventions are not possible anymore for $n \to \infty$, as $\eta(\cdot, \zeta) \le \zeta$. For $\ell > 0$, the (deterministic) explicit solution of

$$dC_t = (r_B(X_t)C_t + r_L\ell)\, dt, \qquad C_{t_n} = c_n$$

converges to $-\ell$ for $n \to \infty$ in arbitrarily short time by assumption $r_B > r_L$, leading to $g(\ell, -\ell, x_3)$ as payout. Possible loan defaults would not change this result, and lead to the same payout $g(S^c) \equiv U(0)$. For $\ell = 0$, the boundedness of $r_B(x)$ for $x_2 > 0$ proves the result.

Finally, we have to find a nonnegative function w as strict supersolution that increases faster than v for $|x| \to \infty$, e.g., super-linearly in view of Prop. 4.3.2. As first criterion, this w has to satisfy $\sup_{\beta \in \{0,1\}}\{w_t + \mathcal{L}^\beta w\} \le -\kappa$ for a $\kappa > 0$ in $[0, T] \times S$. Consider for some $b > 1$, $a > 0$, and a $\tilde{\kappa}$ to be specified:

$$w(t, x) := \exp(-\tilde{\kappa}t)\left(1_{bx_1+x_2\ge 0}(bx_1 + x_2)^2 + bx_1 + x_2 + a\right)$$

(C^1 continuity is sufficient, we do not need to consider a suitably smoothed version). Its generator on S has the following form:

$$\exp(\tilde{\kappa}t)\mathcal{L}^\beta w = C_1x_1^2 + C_2x_2^2 + C_3x_1x_2 + C_4x_1 + C_5x_2 + C_6 + x_2(2x_2 + 2bx_1 + 1)r_B(x) \quad (4.12)$$

for suitable constants $C_i = C_i(x_3, \beta) \in \mathbb{R}$. Note that $bx_1 + x_2$ is a norm on the cone $\{x_1 + x_2 \ge 0, x_1 \ge 0\}$, as is easily checked. By equivalence of all norms, we see that the first part in (4.12) (without r_B) can be bounded by $C\left(1 + (bx_1 + x_2)^2\right)$. For $\{x_2 < 0\} \cap S$, the factor $x_2(2x_2 + 2bx_1 + 1)$ in front of r_B is negative, so that $\tilde{\kappa}$ only needs to depend on $\sup_{x_2>0,x_3} r_B(x)$ and other constants to achieve our desired goal.

One checks easily that thanks to the fixed costs, $w - \mathcal{M}w \ge \kappa$ in $[0, T] \times \mathscr{S}$ (for another $\kappa > 0$), provided that $\eta(\cdot, \zeta) \le b\zeta$ (i.e., an impulse can increase the equity value at most by a factor of $b - 1$).
The function w as defined above does not yet satisfy $w(t, x) \to \infty$ for $|x| \to \infty$ on S^c, the complement of S. However, we have to take care that modifying w on S^c does not negatively affect the property $w - \mathcal{M}w \ge \kappa$ in $[0, T] \times \mathscr{S}$. The idea is to adapt the impulse function Γ on S^c so that impulses go in the direction of a minimum point $(0, -p, x_3)$ with $p > 0$ (for each fixed economy), and to introduce a function $K \le 0$ in the problem formulation to take care of the fixed costs, with

$$\mathcal{M}w(t, x) = \sup_{\zeta \in \{0,\ldots,x_1\}} \{w(t, \Gamma(x, \zeta)) + K(t, x, \zeta)\}; \quad (4.13)$$

correspondingly, the sum of the fixed costs K over all impulses effected is added in the objective function. All these changes in Γ and K do not affect the value function, because it is impossible to get back to S once S^c is reached, and thus the value function is constant on S^c. What is more, for any starting point $x \in \mathscr{S}$, we may wlog modify the trajectory $\Gamma(x, \{0, \dots, x_1\}) \cap S^c$ because it is never optimal to jump to S^c. We define on S^c the function

$$\tilde{w}(t, x) := \kappa_1(t)|(x_1 - 0, x_2 - (-p))| + \kappa_2(t)$$

a function whose contour lines form concentric circles around the point $(0, -p)$. We choose $\kappa_2(t)$ such that $\tilde{w}(t, (0, -p, x_3)) = w(t, (0, -p, x_3))$. We take as new function $\hat{w} := \max(w, \tilde{w})$ in a suitably smoothed form. For a small enough $\kappa_1 > 0$, the intersections where $w = \tilde{w}$ are curves completely within the interior of S^c (see Figure 4.2). Denote $R := \{x : \tilde{w}(t, x) > w(t, x)\} \subset S^c$, and $NR := \{x : \tilde{w}(t, x) < w(t, x)\}$. We define the distance to R for each x_1 separately as $d(t, x, R) := \max(x_2 - \sup_{y \in R, y_1 = x_1} y_2, 0)$. For any starting value $x \in NR$,

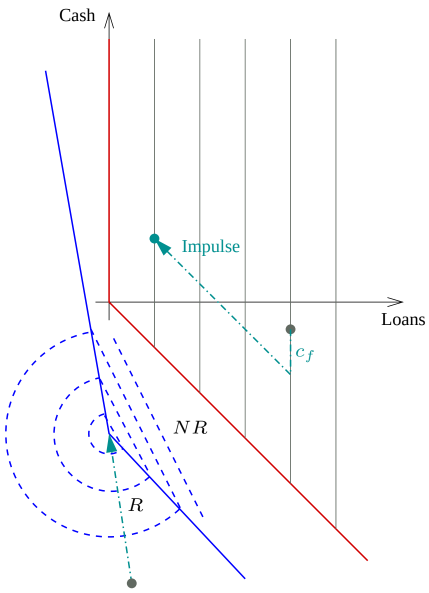

Figure 4.2: Contour lines of w, $\tilde{w}$ in the proof of Theorem 4.3.3 in a (L, C) graph for fixed economy

we modify the trajectory of Γ on S^c such that $\Gamma(x, \{0, \dots, x_1\}) \subset NR$, while still respecting $w(t, \Gamma(x, \zeta)) \leq w(t, x)$. For $x - (0, c_f, 0)^T \notin NR$ (which we can take wlog in S^c), we start modifying K:

$$K(t, x, \zeta) := \min(-c_f + d(t, x, R), 0)$$

For any starting point $x \in R$ with $x_1 > 0$, we set

$$\Gamma(x, \zeta) := \frac{x_1 - \zeta}{x_1} x + \frac{\zeta}{x_1} (0, -p, x_3)^T,$$

such that impulses go towards $(0, -p, x_3)$ and the impulse direction is perpendicular to the contour lines of $\tilde{w}$; if $x_1 \leq 0$, changing Γ is not necessary. Wlog, we can choose Γ continuous on $\overline{R} \cap \overline{NR}$ because of the modification of its trajectory in $NR \cap S^c$. We conclude that with the modified Γ, K and $\mathcal{M}$ as defined in (4.13), $\hat{w} - \mathcal{M}\hat{w} \geq \kappa$ in $[0, T] \times \mathscr{S}$ for some $\kappa > 0$.

The inequality $\hat{w} - g \geq \kappa$ holds in $[0, T] \times S^c$ if we set the constants in w, $\tilde{w}$ large enough. $\square$

4.4 Frictionless markets

We investigate in this section stochastic control models related to our original model that help us to understand better the model in a few special cases. Without transaction costs (i.e., $\eta(\cdot,\zeta) \equiv \zeta$ and $c_f = 0$), the model can be reduced in dimension, and the controls boil down to one scalar control variable representing the leverage of the bank. If we define $\pi_t := \frac{L_{t-}}{L_{t-}+C_t}$, then the dynamics for the equity value $Y_t := L_t + C_t$ reads as follows:

$$\begin{aligned} dY_t &= -\delta(M_{t-})dN_t + (r_B(X_t)C_t + r_L L_t)\,dt \\ &= -\delta(M_{t-})dN_t + ((1-\pi_t)r_B(\pi_t Y_t, (1-\pi_t)Y_t, M_t) + \pi_t r_L)\,Y_t dt \end{aligned} \tag{4.14}$$

Note that in the original model setting, π_t cannot be chosen freely by the controller: While it is possible to reduce immediately π_t to 0 (impulses in the original model), the possible increase $\Delta\pi_t$ in time depends on the "new business arrival process" P, and the previous leverage π_t. To obtain meaningful results, we leave all these restrictions aside, and analyze the Hamilton-Jacobi-Bellman (HJB) equation of stochastic control for $\pi \in [0,K]$ for some $K > 0$:[3]

$$\begin{aligned} \sup_{\pi\in[0,K]} \{u_t + \mathcal{L}^\pi u\} = 0 \qquad & \text{in } S_T \\ u = g \qquad & \text{in } \partial^+ S_T \end{aligned} \tag{4.15}$$

where this time, $S = (0,\infty) \times \{0,1\}$, and the infinitesimal generator $\mathcal{L}^\pi$ on S has the form

$$\begin{aligned} \mathcal{L}^\pi u(y,m) = {} & (u(y-\delta(m),m) - u(y,m))\,\lambda(m)\pi y \\ & + (u(y,1-m) - u(y,m))\,\lambda_{m,(1-m)} + ((1-\pi)r_B(\pi y,(1-\pi)y,m) + \pi r_L)y u_y(y,m). \end{aligned}$$

If (4.15) has a suitably differentiable solution, then verification results say (see, e.g., Øksendal and Sulem [93], Theorem 3.1) that this solution is equal to the value function, and a maximizer in (4.15) yields an optimal stochastic control. Let us assume that this is the case for our stochastic control value function $\tilde{v}$, and for simplicity that r_B is constant. Then for $\delta = \delta(m)$, $\lambda = \lambda(m)$

$$\pi \mapsto \pi y\,[(\tilde{v}(t,y-\delta) - \tilde{v}(t,y))\lambda + (r_L - r_B)v_y(t,y)]$$

has to be maximized (separately for each m), with the solutions

$$\hat{\pi} = \begin{cases} 0 & \text{if } (\tilde{v}(t,y-\delta) - \tilde{v}(t,y))\lambda + (r_L - r_B)\tilde{v}_y(t,y) < 0, \\ K & \text{if } (\tilde{v}(t,y-\delta) - \tilde{v}(t,y))\lambda + (r_L - r_B)\tilde{v}_y(t,y) > 0, \\ [0,K] & \text{else.} \end{cases}$$

Which of the conditions is satisfied, depends very much on the boundary values in S^c and on their propagation inside S.

Our above analysis shows that quite trivial optimal controls (either no loans, or highest possible leverage) can be expected in this simple setting; these results are confirmed in §4.5. More interesting results can be expected if we introduce a risk-dependent refinancing function for r_B, as done in our model. If r_B depends only on the leverage (in our case π), then one can derive criteria r_B has to satisfy to ensure that the maximum in (4.15) is attained in $(0,K)$.

We would like to stress that these trivial stochastic controls were derived under several assumptions (notably the smoothness of the corresponding value function), and are only optimal if π

[3] K may be interpreted as some upper bound regulators impose on the bank's leverage.

can be set in an arbitrary way in the interval $[0, K]$. In general (including the Markov-switching economy, and restrictions on the control process π), the picture it not so clear anymore: There may be an incentive to keep loans although they are not profitable in the momentary economic situation. In the impulse control case, this can be observed in the numerical results of §4.5.

Large portfolio approximation

Next, we consider an approximation with an infinitely granular portfolio. In the limit, the randomness related to individual defaults disappears, and the economy process M remains the only risk factor. If we increase for constant loan nominal the portfolio granularity to $n \in \mathbb{N}$, then loan defaults are more frequent, but have a smaller proportional effect. This is reflected in the generator of the n-granular SDE (for $\tilde{x} = (x_1, x_2)$):

$$\mathcal{L}^{n,\beta}u(x) = \left(u(\tilde{x} + \frac{1}{n}\begin{pmatrix} -1 \\ 1-\delta(x_3)\end{pmatrix}, x_3) - u(x)\right)\lambda(x_3)nx_1 + \left(u(\tilde{x} + \frac{1}{n}\begin{pmatrix}\beta \\ -\beta\end{pmatrix}) - u(x)\right)\lambda_P n$$
$$+ \left(u(\tilde{x}, 1-x_3) - u(x)\right)\lambda_{x_3,(1-x_3)} + (r_B(x)x_2 + r_L x_1)u_{x_2},$$

For $u \in C^1(S)$, the generator $\mathcal{L}^{n,\beta}u$ converges uniformly on each compact for $n \to \infty$ to:

$$\mathcal{L}^{\infty,\beta}u(x) = -\lambda(x_3)x_1 u_{x_1}(x) + (1-\delta(x_3))\lambda(x_3)x_1 u_{x_2}(x) + \beta\lambda_P u_{x_1}(x) - \beta\lambda_P u_{x_2}(x)$$
$$+ \left(u(\tilde{x}, 1-x_3) - u(x)\right)\lambda_{x_3,(1-x_3)} + (r_B(x)x_2 + r_L x_1)u_{x_2}(x),$$

Following Jacod and Shiryaev [66], ch. IX.4, this implies the weak convergence in law of the n-granular SDE solution of (4.1.L), (4.1.C) to the solution of

$$\begin{aligned} dL_t^\infty &= (-\lambda(M_t)L_t^\infty + \beta_t\lambda_P)\,dt, \qquad L_0^\infty \in \mathbb{R}_0^+ \\ dC_t^\infty &= (r_B(X_t^\infty)C_t^\infty + r_L L_t^\infty + (1-\delta(M_t))\lambda(M_t)L_t^\infty - \beta_t\lambda_P)\,dt \end{aligned} \tag{4.16}$$

with the still unchanged Markov switching process M. Here, the dynamics for the equity value $Y_t^\infty := L_t^\infty + C_t^\infty$ with controlled leverage π (and no transaction costs) is:

$$dY_t^\infty = (-\pi_t\delta(M_t)\lambda(M_t) + (1-\pi_t)r_B(\pi_t Y_t^\infty, (1-\pi_t)Y_t^\infty, M_t) + \pi_t r_L)\,Y_t^\infty dt \tag{4.17}$$

If we assume that we are able to control freely the leverage $\pi \in [0, K]$, the corresponding HJB equation is again (4.15), but with the infinitesimal generator $\mathcal{L}^\pi$ equal to

$$\mathcal{L}^\pi u(y,m) = (-\pi\delta(m)\lambda(m) + (1-\pi)r_B(\pi y, (1-\pi)y, m) + \pi r_L)\,y u_y(y,m)$$
$$+ \left(u(y, 1-m) - u(y,m)\right)\lambda_{m,(1-m)}.$$

It is clear from (4.17) that the optimal strategy is obtained by maximizing the instantaneous return for each economy state separately. Under the assumption that $r_B = r_B(\cdot, m)$ only depends on the leverage π (and on the economy state), the instantaneous return R to maximize is ($\delta = \delta(m)$, $\lambda = \lambda(m)$)

$$R(\pi) := -\pi\delta\lambda + (1-\pi)r_B(\pi) + \pi r_L$$

with derivative

$$R'(\pi) = -\delta\lambda + r_L - r_B(\pi) + (1-\pi)r_B'(\pi).$$

We assume for the moment $r_B \in C^2$, $r_B \geq \rho$, $r_B \equiv \rho$ on $\{\pi \leq 1\}$, $r_B'(\pi) > 0$ on $\{\pi > 1\}$ and $r_B'(\pi) > \delta$ on $\{\pi > 1+1/\delta\}$ for some $\delta > 0$. Then the maximizer $\hat{\pi}$ is 0 if loans are not profitable

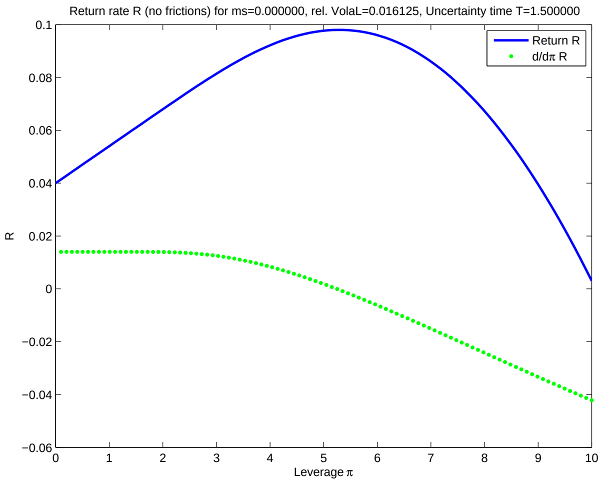

Figure 4.3: Return rate R (solid line) and R' (dotted) dependent on leverage π in large portfolio approximation. Example parameters are as used in §4.5 for expansion: $\delta = 1$, $\lambda = 0.026$, $r_L = 0.08$, $\rho = 0.04$. The variable refinancing rate r_B is based on a Vasicek loss distribution with default probability $p = 1.5\delta\lambda$, correlation $\varrho = 0.2$ and $LGD = 0.4$ (PD according to Example 4.2.2, form (b))

on average ($r_L - \rho - \delta\lambda < 0$). If loans are profitable ($r_L - \rho - \delta\lambda > 0$), then there is a maximizer $\hat{\pi} \in (1, \infty)$, and $R'(\hat{\pi}) = 0$. Typical R and R' are shown in Figure 4.3 for r_B as proposed in §4.2.2, Example 4.2.2.

As the maximal rate of return R is deterministic for each state of the economy, the corresponding value function then depends only on how long the bank spends in each state of the economy until T. An explicit representation of the value function in form of a matrix exponential can then be given using the results in the appendix (§A.3). The reader will certainly agree that the assumption of being able to manipulate freely the proportion of loans in the bank's portfolio is quite unrealistic. In reality, issuing loans will be a slow process, and reducing loan exposure may be quick, but costly — so we would expect some sort of interplay between the economic states. We see that transaction costs and/or control restrictions are keys to a good model, because otherwise the result can be as unrealistic as for the large portfolio approximation. Furthermore, the large portfolio approximation shows that the discreteness of our portfolio is necessary to have risk other than economy switching.

4.5 Numerical results

This section starts with a short description of the numerical scheme used to solve the PDE (4.9) and thus the combined impulse and stochastic control problem. Then, numerical results are presented and discussed from an economic point of view.

4.5.1 Finite Difference scheme

We have discussed already in Chapter 3 how an HJBQVI can be solved using iterated optimal stopping. Our algorithm used here is very similar to the one proposed in §3.5.1.

Computations were carried out in MATLAB. The initial PDE iteration and the optimal stopping problems are solved using a finite difference scheme on a rectangular space grid. The optimal stochastic control to use in each time step is calculated using the value function from the previous timestep (explicit), the rest of the timestepping is done in a θ-scheme with $\theta = 0.5$ (Crank-Nicolson); see §3.5.1, or for instance [102], [108] for details. The (discrete) optimal stopping problem is solved using PSOR (projected successive over-relaxation) with adaptive relaxation parameter. We used a bespoke optimization routine for the impulse maximization; the destination of a potential impulse from (t, x) after n iterations is determined by the maximizer in $\mathcal{M}v^{n-1}(t, x)$. To handle boundary values at ∞, the computational domain was enlarged in all iterations, and Neumann boundary values equal to the derivative of the discounted utility function were applied at the cutoff boundary.

The optimal impulses after n optimal stopping iterations shown on the following pages follow the rule "Jump in points where $v^n \leq \mathcal{M}v^{n-1}$" (perhaps infinitely often). This can be justified from the discussion in §3.4.4.

4.5.2 Numerical examples

In the numerical examples we used the following parameter values: The utility function is of CRRA type (constant relative risk aversion) and given by $U(x) = \sqrt{x}$. The Markov chain intensities for the economy are set to be equal to $\lambda_{10} = \lambda_{01} = 0.3$, the default intensities per loan are 2.6% in expansion and 4.7% in contraction (which seems to be a rather conservative estimate for the changes between different economic states), with no loan default recovery ($\delta \equiv 1$). The risk-free rate ρ is constant 0.04, the loan interest rate is set to 0.08. We used the finite variable refinancing cost in §4.2.2, Example 4.2.2 with $LGD = 0.4$ and PD (b), based on a Vasicek loss distribution with $p = 1.5\lambda$, and correlation $\varrho = 0.2$ (0.4) in expansion (contraction). The resulting r_B is the green dotted line in Figure 4.13. (This refinancing cost may seem relatively high, however this reflects that our bank's only assets are risky loans.) The fixed transaction cost was 0.5, while the market value of securitized loans was chosen according to form (a) (strongly procyclical form; see §4.2.2, Example 4.2.1), which results in proportional transaction costs of 0% (about 6.5%) in expansion (contraction).

It will be shown below in Figure 4.12 that the stochastic control variable β has only a small impact on the value function and on the optimal impulse control strategy; unless stated otherwise, we will therefore take $\beta \equiv 0$.

The first Figure 4.4 shows the value function, i.e., the expected terminal utility under optimal impulses, for start in expansion or in contraction. Figure 4.5 demonstrates the benefit of controlling the loan exposure: the utility indifference graph shows the cash value of impulses (compare also Figure 4.6). At the risk of oversimplification, this quantity can be interpreted as the maximum salary the bank should pay its risk manager for implementing the optimal impulse strategy (compared to no impulses at all). For our chosen parameters, this benefit is greater in good economic times and reaches up to about 10% of the loan exposure. This is mainly due to the lower proportional transaction costs during expansion.[4] The cash value of impulses is lower in recession, simply because the essence of the optimal strategy is to wait for the next boom (compare Figure 4.7 and explanation below).

The form of the optimal impulse control strategy is depicted in Figures 4.7 and 4.8. Again we

[4] If the market value is chosen according to form (b), the benefit is smaller in expansion, because then proportional transaction costs are low for both economic states.

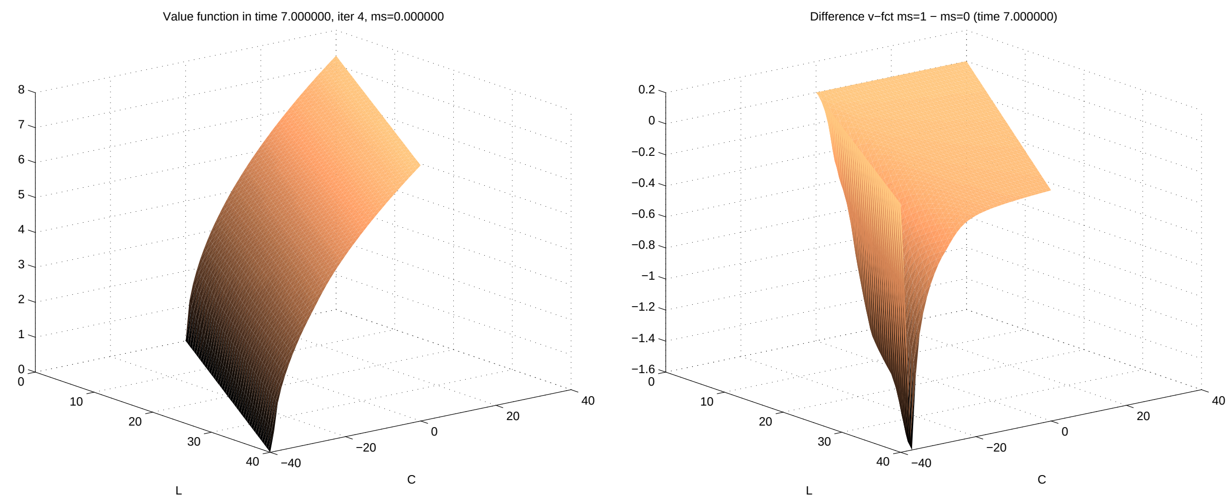

Figure 4.4: Value function in expansion time (left) and difference value function in contraction minus value function in expansion (right), for $T = 7$. The x coordinate is the loan exposure, the y coordinate the cash. Parameters for this example are as described in the text. The timestep for this numerical simulation was 0.5 years, 5 optimal stopping iterations were carried out

see that securitization in good times is more beneficial than in bad times, essentially because the high proportional transaction costs of around 6.5% in contraction keep the bank from acting near $\{x + y = 0\}$. This is remarkable as the high leverage and the resulting default risk and refinancing costs endanger the bank's existence. In such a situation, it is optimal for the bank to wait for better times. This lack of admissible impulses is compensated in better economic times: here the loan exposure of the problematic region is reduced to practically 0 as a provision for contraction, which amounts to a (temporary) liquidation of the bank. It is instructive to compare this result to the "only expansion" case in Figure 4.10, where such a provision is unnecessary. If the market value of loans is less procyclical, then a lot less interventions take place in expansion, and more in contraction; this can be seen in the right column of Figure 4.14, where different function choices for refinancing cost and market value are compared. We can conclude that (proportional) transaction costs seem to be a crucial input into our model.

In the impulse graphs dependent on time to maturity (Figure 4.8; only in expansion), we see that immediately prior to T, the fixed transaction costs make it optimal to wait rather than to transact. For comparison, we have also included the same impulses-over-time graph for lower fixed transaction costs $c_f = 0.2$ (Figure 4.9). We observe in Figure 4.9 that for $T = 3$, the transaction region is larger than for $T = 1$ — here the fear that at terminal time the bank may end up in contraction with the corresponding low liquidation value of loans dominates the desire to get a higher return rate until terminal time, and dominates also the reluctancy to pay the (fixed) transaction costs now. We note that naturally, this last effect does not affect impulses in contraction.

Further findings from our analysis and numerical results for our chosen set of parameters are:

(i) A bank has an incentive to reduce a high leverage — the higher the refinancing cost r_B or the loan default rate λ, the stronger the incentive. On the one hand, this can be inferred from the analysis in frictionless markets (§4.4). For the original setting, a comparison of the different function choices for refinancing cost and market value can be found in Figure

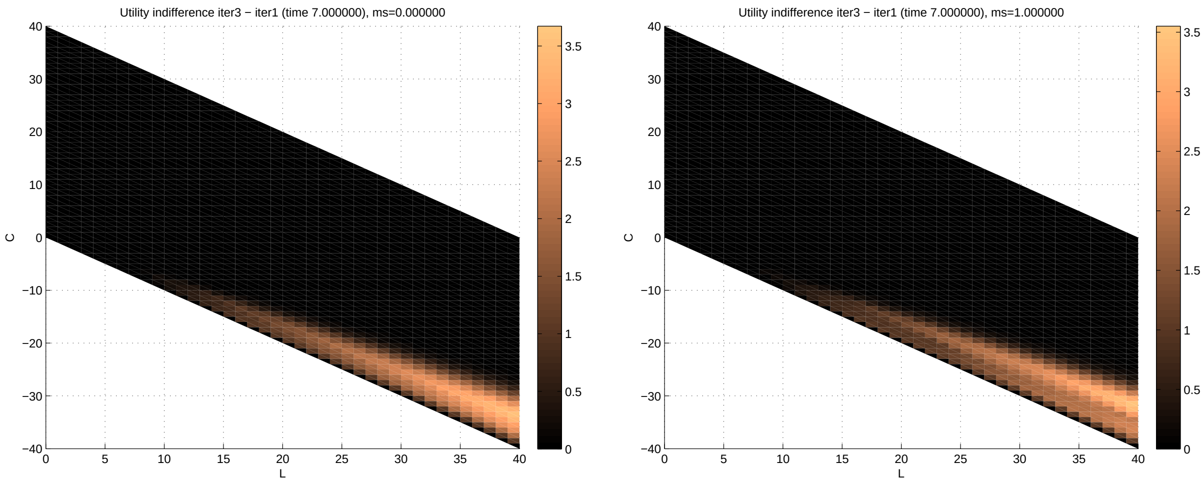

Figure 4.5: Cash value of impulses in expansion (left) and contraction (right) for $T = 7$ (in a bird's view; height according to colour code on the right). Shown for each point x is the value a such that $v_3(x_1, x_2 - a) = v_1(x_1, x_2)$ (v_3 being the value function with impulses, v_1 without), i.e., the cash the impulse-controlled bank can pay out while still being better off than the uncontrolled bank in the same situation. The cash value of impulses is practically 0 in the large dark region, and it is maximal in the lower right corner. Same data as in Figure 4.4

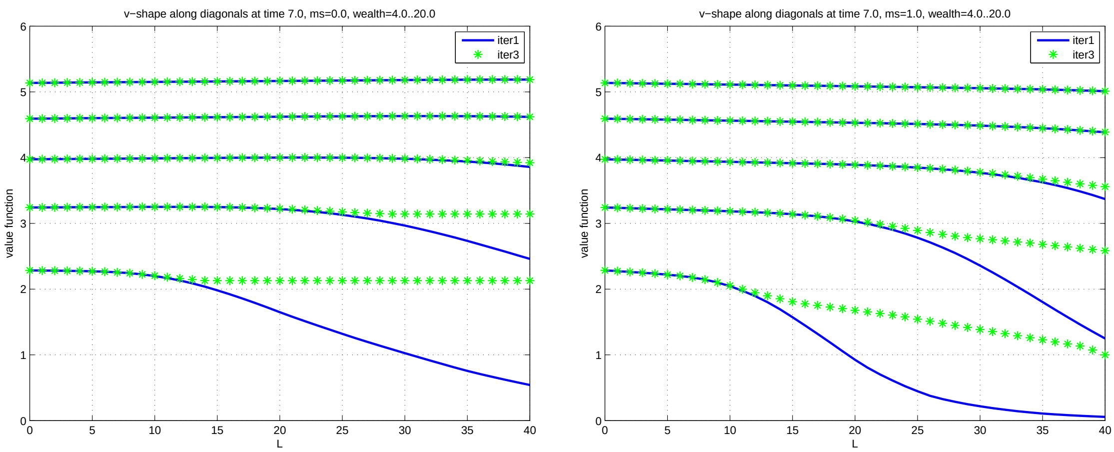

Figure 4.6: Value function with (green stars) and without impulses (blue line) in expansion (left) and contraction (right) for $T = 7$. Each (dotted or solid) line shows the value function along a diagonal with constant equity capital $L + C$. In the above graph, the x coordinate is the loan exposure; here, a high loan exposure value corresponds to a high leverage of the bank. Roughly, an impulse is optimal when the starred line is significantly above the solid line — the larger the difference, the more valuable an impulse is. The starred line in expansion is in large parts constant because there impulses always end in $L = 0$, independent of the starting point; this line is decreasing in contraction essentially because the proportional transaction costs lead to a different equity capital level. The selected equity capital levels range from 4 to 20. Same data as in Figure 4.4

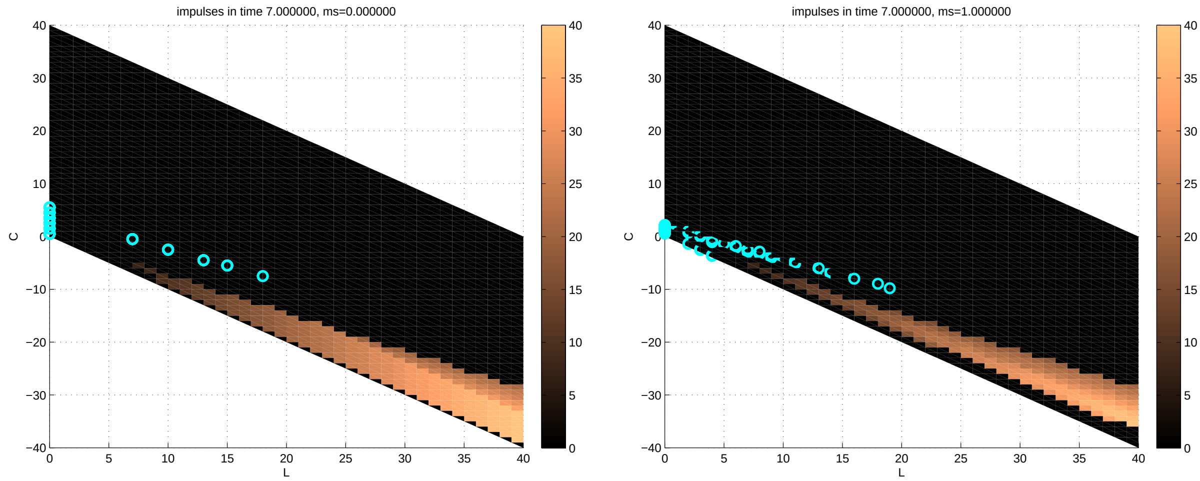

Figure 4.7: Optimal impulses in expansion (left) and contraction (right) for $T = 7$. The light areas mark the impulse departure points (with the lightness indicating how far to the left the impulses goes, i.e., how many loans are sold), the cyan circles represent the corresponding impulse arrival points. There are no impulses in the large dark region, and the largest impulses occur in the lower right corner. Same data as in Figure 4.4

4.14.

(ii) In expansion, the loan exposure should be reduced to 0 if the initial leverage is sufficiently high (see Figure 4.7) — this hinges on the absence of proportional transaction costs in our expansion case, and also on the high transaction costs in recession (compare to right column of Figure 4.14). Positive proportional transaction costs, on the other hand, lead to significantly smaller impulses, as can be seen from the right hand side (contraction) of Figure 4.7.

(iii) In the example complete with stochastic control, our results (Figure 4.12) indicate that building up loan exposure in a recession is not optimal. Note however that the loan return rate r_L is constant and so does not reflect the additional risk the bank takes when issuing loans during contraction.

(iv) For reasonably profitable loans, it is optimal to issue loans also in contraction, as Figure 4.11 shows (although r_L is not risk-adjusted). This holds in particular if building up loans takes a relatively long time, as determined by the “new busines arrival” intensity λ_P. Note however that the new busines in contraction is on a much smaller scale than in expansion because of transaction costs — the stochastic control cannot be so easily undone.

(v) We note (without graph) that the impact of a different / more risk-averse utility function is not substantial (even if it is the log utility function).

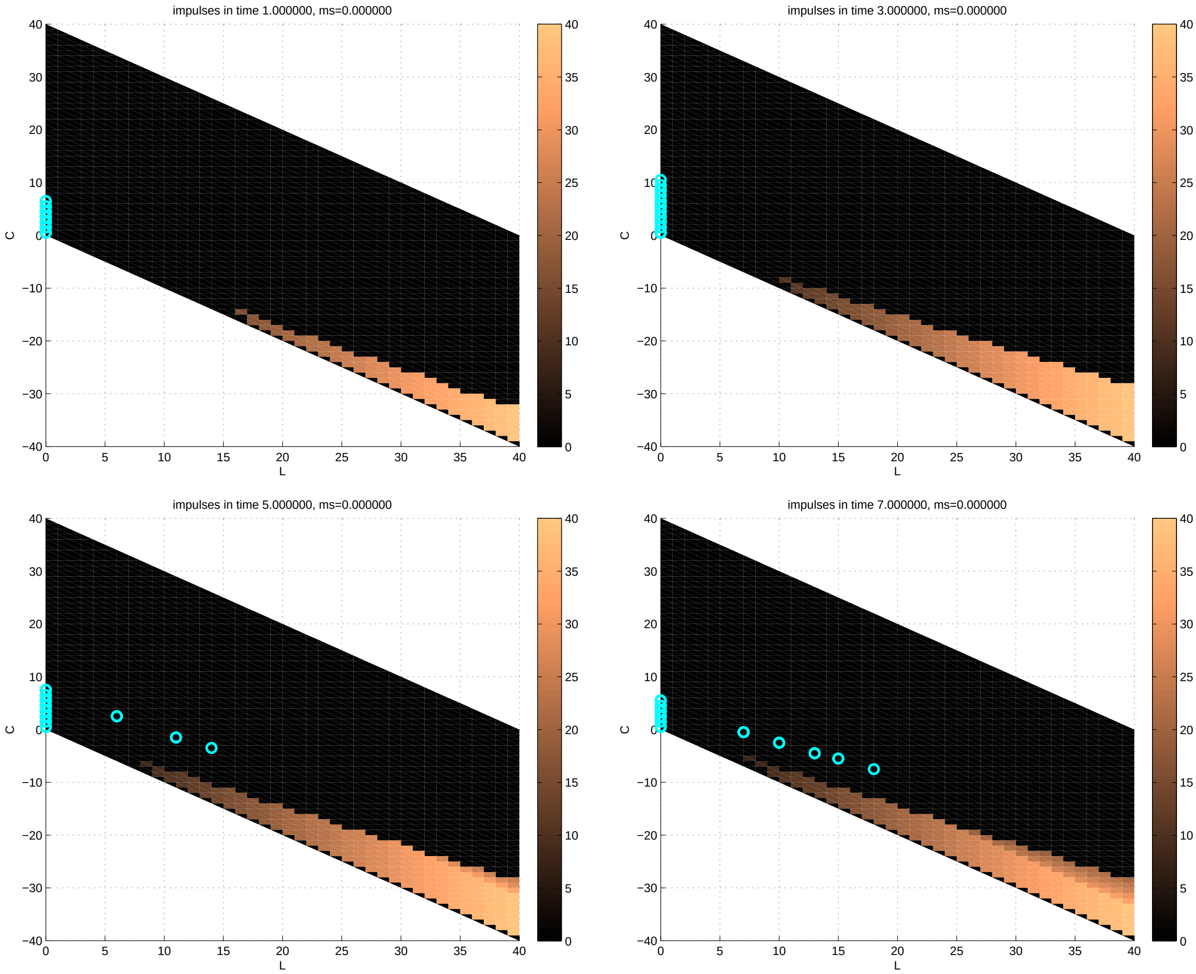

Figure 4.8: Optimal impulses in expansion for different T: top left $T = 1$, top right $T = 3$, bottom left $T = 5$ and bottom right $T = 7$. For the colour code, see the explanations in Figure 4.7. Same data as in Figure 4.4

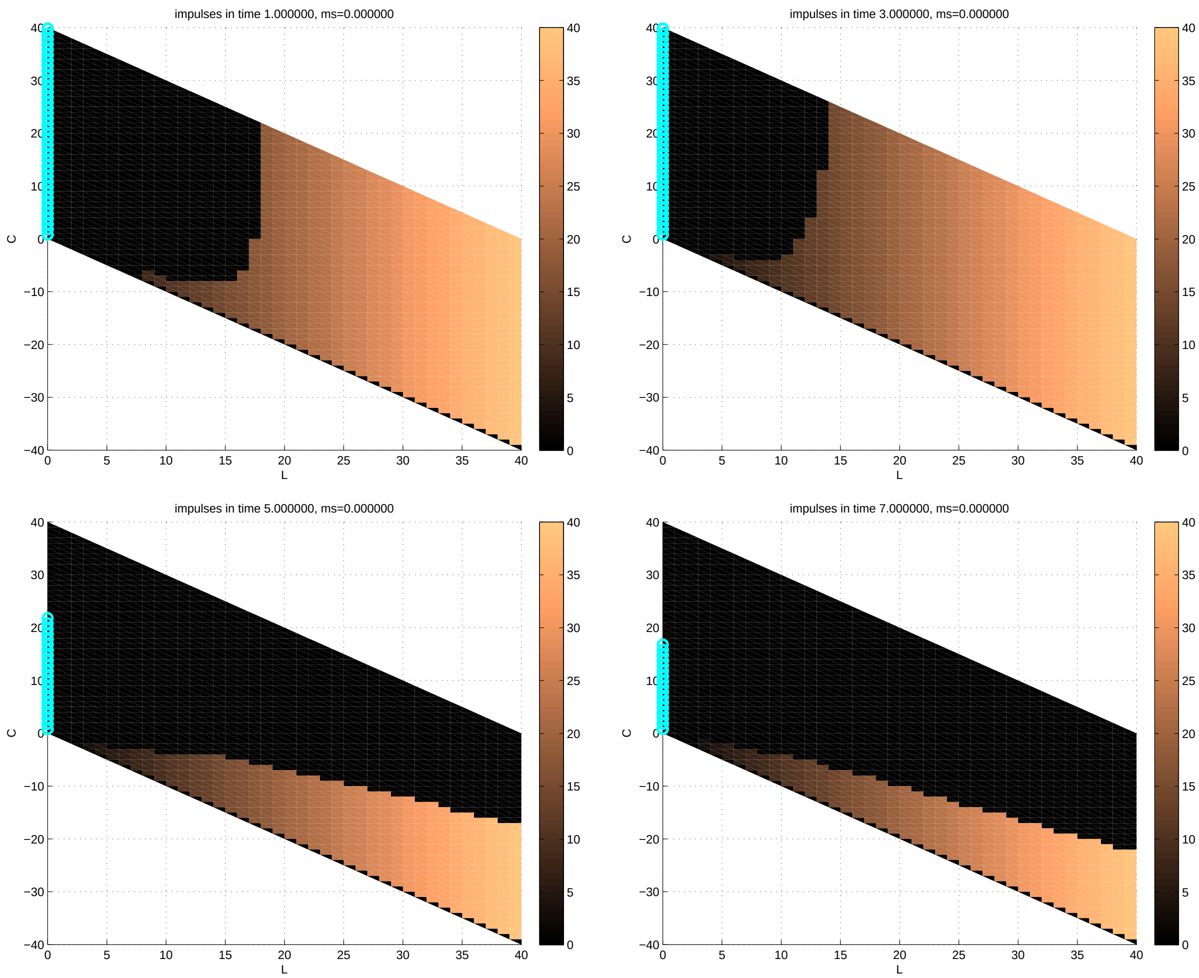

Figure 4.9: Optimal impulses in expansion for different T, fixed transaction costs $c_f = 0.2$: top left $T = 1$, top right $T = 3$, bottom left $T = 5$ and bottom right $T = 7$. For the colour code, see the explanations in Figure 4.7. Apart from c_f, same data as in Figure 4.4

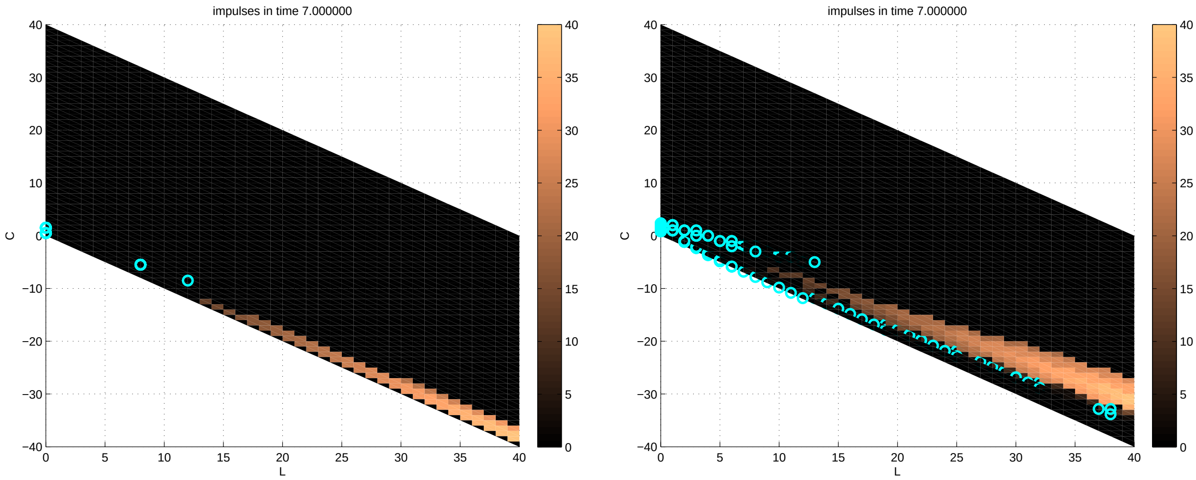

Figure 4.10: Impulses without Markov switching, for only expansion (left) and only contraction (right) for $T = 7$. Market value according to Example 4.2.1, procyclical form (a), corresponding to 0% (about 17%) proportional transaction costs in expansion (contraction). For the colour code, see the explanations in Figure 4.7. Otherwise, same data as in Figure 4.4

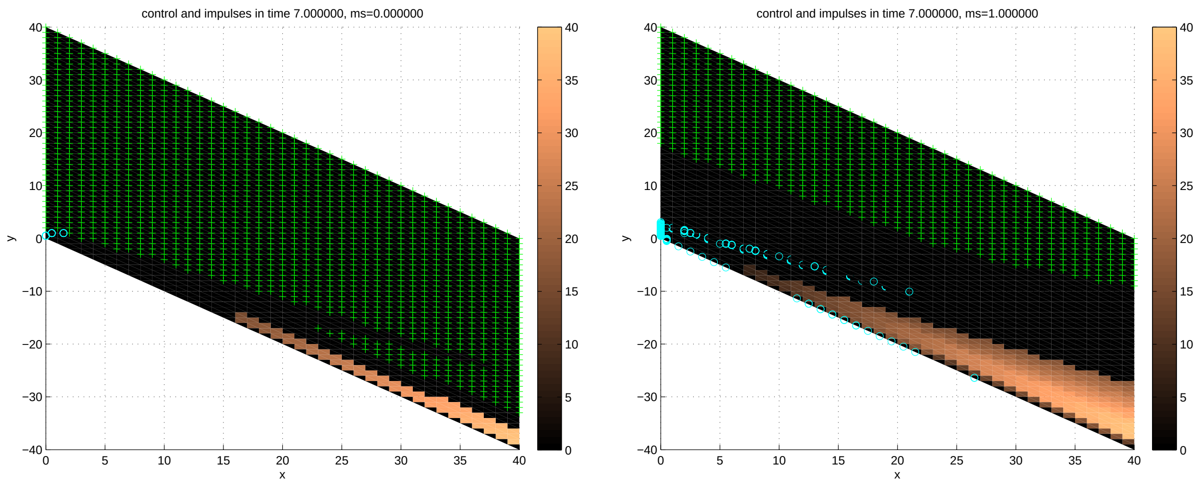

Figure 4.11: Optimal combined impulse and stochastic control for $T = 7$, with default intensities per loan of 2% (4.7%) in expansion (contraction). Market value according to Example 4.2.1, procyclical form (a), corresponding to 0% (about 3%) proportional transaction costs in expansion (contraction). For the colour code, see the explanations in Figures 4.7 and 4.12. Business arrival intensity $\lambda_P = 2$, otherwise same data as in Figure 4.4

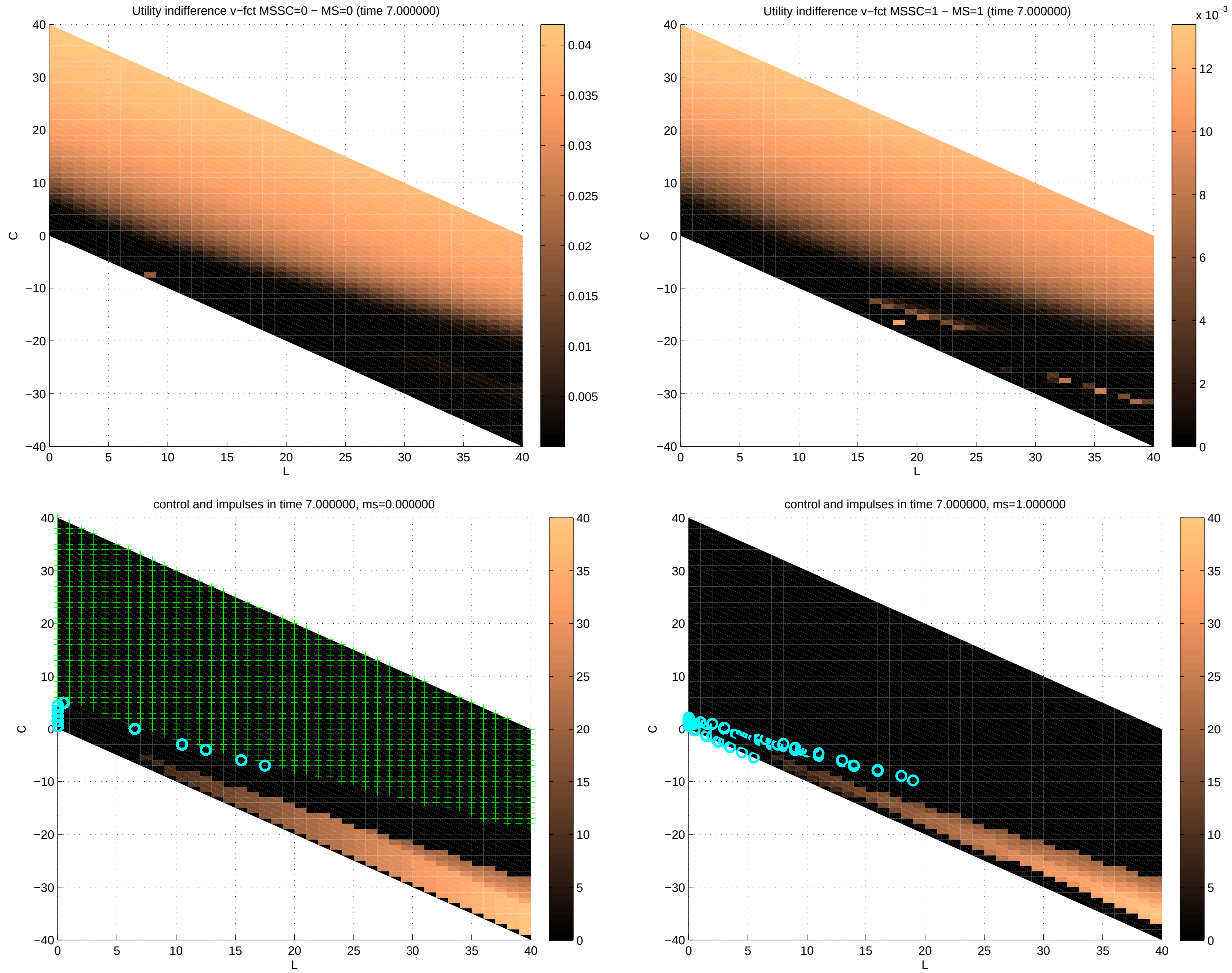

Figure 4.12: Impulse and stochastic control: Cash value of additional stochastic control (top row), and optimal strategy (bottom row) in expansion (left) and contraction (right), for $T = 7$. The cash value shows (as in Figure 4.5) the value a such that $v_{SC}(x_1, x_2 - a) = v(x_1, x_2)$ (v_{SC} being the value function including stochastic control, v only with impulse control). The impulses are plotted in the same way as in Figure 4.7, points with positive stochastic control are marked with a green + (green light for customers). Business arrival intensity $\lambda_P = 2$, otherwise same data as in Figure 4.4

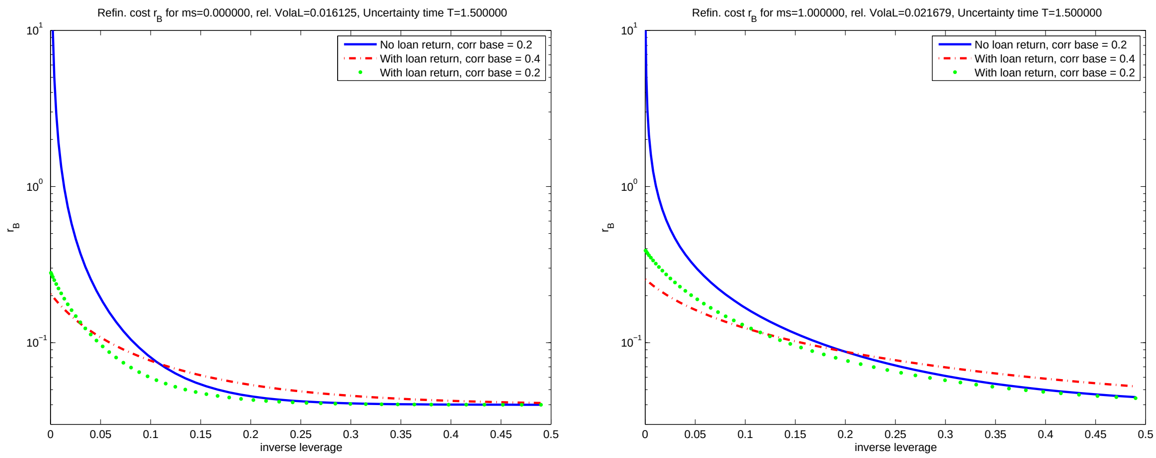

Figure 4.13: Variable refinancing rates in expansion (left) and contraction (right) used in Figure 4.14, based on a Vasicek loss distribution with default probability $p = 1.5\delta\lambda$ and $LGD = 0.4$. The rates are shown as a function of the inverse leverage $\frac{L+C}{L}$ in a logarithmic scale. The green dotted line is r_B with loan return (PD according to Example 4.2.2, form (b)) and correlation $\varrho = 0.2$ (0.4) in expansion (contraction); the red dash-dotted line shows the same r_B for ϱ increased by 0.2; the blue line is r_B according to form (a) (infinite at $\{x+y=0\}$) for correlation $\varrho = 0.2$ (0.4)

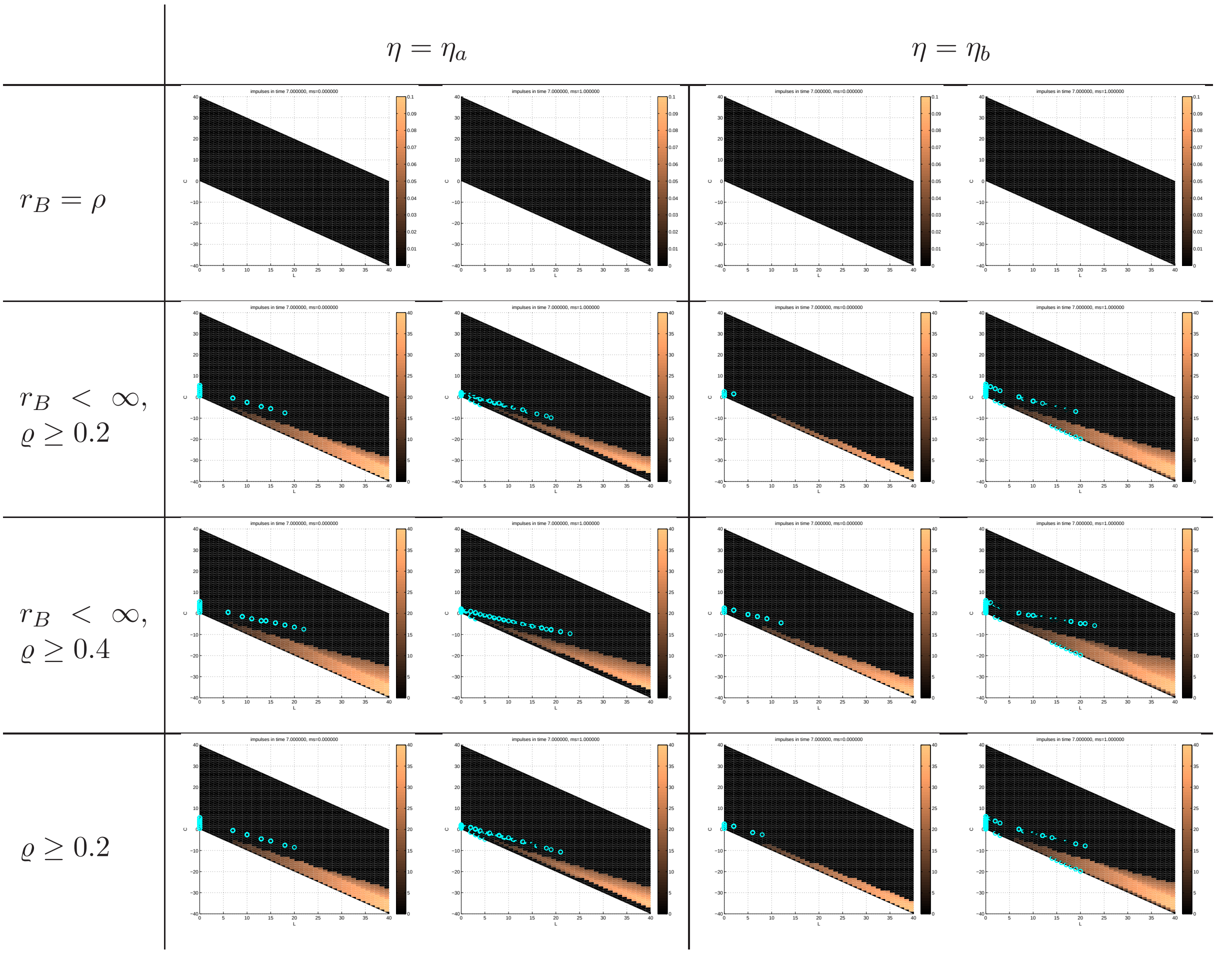

Figure 4.14: Optimal impulses for different refinancing functions (rows) and different market values (columns) for $T = 7$. In each cell, impulses in expansion are on the left, and impulses in contraction on the right. Refinancing cost from top to bottom: (1) r_B equal to the risk-free rate ρ; (2) r_B based on Vasicek loss distribution with loan return (Example 4.2.2, form (b)) for $p = 1.5\delta\lambda$ and correlation $\varrho = 0.2$ (0.4) in expansion (contraction); (3) r_B with the same form (b), but correlation $\varrho = 0.4$ (0.6); (4) r_B according to form (a) (infinite at $\{x + y = 0\}$) for correlation $\varrho = 0.2$ (0.4). Market values from left to right: (1) Market value η according to Example 4.2.1, form (a) (procyclical), corresponding to no proportional transaction costs in expansion, and $\approx 6.5\%$ in contraction; (2) market value η according to form (b), corresponding to about 0% (1.7%) proportional transaction costs in expansion (contraction). Otherwise, same data as in Figure 4.4

Chapter 5

Summary and conclusions

This thesis centres on combined stochastic and impulse control and applications, its relation to Hamilton-Jacobi-Bellman quasi-variational inequalities and the numerical solution of such HJBQVIs.

We started with a short introduction to the theory of viscosity solutions for PDEs, where we discussed also extensions to equations with integral term.

In **Chapter 2**, we showed existence and uniqueness of viscosity solutions of HJBQVIs: The value function of combined stochastic and impulse control was shown to be a viscosity solution using stochastic means, while for the uniqueness part a comparison theorem was proved using purely analytical techniques. The results we obtained are quite general, and the (minimal) assumptions – basically (local Lipschitz) continuity, continuity of the value function at the boundary, compactness and "continuity" of the transaction set, and existence of a strict viscosity supersolution – are sufficient to guarantee a continuous solution on $\mathbb{R}^d$. We note that the Lipschitz continuity assumptions are already needed to ensure existence and uniqueness of the underlying SDE.

The complications to be overcome were mainly: the discontinuous stochastic process and definition of the value function on $\mathbb{R}^d$; the possibly singular integral term in the PIDE, arising from the Lévy jumps; the additional stochastic control.

It is our hope that the parabolic and elliptic results presented can be used to great benefit in applications of impulse control without the need to go into details of viscosity solutions (as typically the value function in – at least financial – applications will be continuous). The comparison result can also be used to carry out a basic stability analysis for numerical calculations.

Admittedly, our results do not cover all special cases — but quite frequently, one should be able to extend the results of this paper easily. For example, state constraints can be handled with a modified framework, where the continuity inside S in general should still hold; see also [80]. This leaves some room for future research.

Chapter 3 investigated numerical methods for (HJB)QVIs, with a focus on the iterated optimal stopping technique. From the results of Chapter 2 and under similar assumptions, we deduced existence and uniqueness of viscosity solutions for the HJBVI of stochastic control and stopping. We showed the convergence of iterated optimal stopping by using contraction-like properties of the iterated optimal stopping operator Q, and a viscosity solution stability result for HJBVIs. The operator Q has contraction-like properties essentially because there is a strict viscosity supersolution of the HJBQVI to be approximated.

In the process of the convergence proof, we derived an interesting new equivalence of two different optimal stopping iterations for exit time problems. The convergence result permitted us also to state a viscosity solution existence and uniqueness theorem for HJBQVIs under assumptions slightly weaker than in Chapter 2.

The last part of Chapter 3 dealt with the concrete numerical implementation of iterated optimal stopping. We presented an algorithm, sketched a proof of convergence of the discretized scheme, and pointed out several details that have to be taken into account for an implementation.

A new model of credit securitization for a bank in **Chapter 4** served as illustrative example for a combined stochastic and impulse control problem; in the model, the bank can control the leverage of its lending activities by securitization impulses. We proved that the value function is increasing and linearly bounded, and that it is the unique (continuous) viscosity solution of an HJBQVI, using the results from Chapter 2. After investigating some related simplified models of stochastic control, we proceeded to the numerical solution of the HJBQVI, and discussed optimal strategies as well as the form of the value function for several parameter sets.

The – at first view surprising – result is that banks should not securitize in a contraction if transaction costs are high, but rather wait for better times. Either the leverage is too high, and a transaction only worsens the situation, or the leverage is not high enough to justify an intervention. This reluctancy to securitize in contraction is compensated by more impulses in expansion times.

Then why would banks sell loans (or ABS) at such large discounts, as observed in the recent credit crisis? One reason may be that the true value (discounted expected earnings) justified the discount; another may be that there was a high risk of the economic situation worsening — in other words, there could be more economic states than just two (this is a straightforward extension of our model). As well, regulatory aspects have been neglected in our model. If the bank has in its portfolio also assets other than loans, e.g., government bonds with non-zero risk weight, then the conclusions might change under the Basel regulation. This could be another explanation for some recent sell-offs during the credit crisis.

A natural and interesting extension of our model would be to handle loan portfolios dependent on several (economic) factors. As this would increase the dimensionality of the problem, numerical results would be however more difficult to obtain. The conclusions of our study might change also if we admit injections of capital into the bank — here we would have to deal with an additional impulse control.

Appendix A

Appendix

A.1 Dynkin's formula and its variants

For a Markov process X and an appropriately chosen function u, Dynkin's formula takes the form

$$\mathbb{E}[u(X_t)] = u(X_0) + \mathbb{E}\left[\int_0^t \mathcal{L}u(X_s)ds\right], \qquad t > 0, \tag{A.1}$$

where $\mathcal{L}u$ denotes the infinitesimal generator of X applied to u, assuming it exists. Typical assumptions for (A.1) to hold in a general jump-diffusion setting are $u \in C^2$, and the boundedness of the first and second derivatives of u (see Gikhman and Skorokhod [53], Chapter II.2.9). These standard assumptions however are too restrictive for our purposes in Chapter 2. In fact, the above conditions can be relaxed, as we will show in this section.

A.1.1 Poisson random measure integral

The standard way to prove Dynkin's formula is to apply first Itô's formula, and then take expectation to eliminate the integrals with respect to Brownian motion and the Poisson martingale measure. For the fully compensated Poisson random measure $\widetilde{N}(ds,dz) = N(ds,dz) - \nu(dz)ds$ and adapted φ, the integral $\mathbb{E}[\int_0^t \int \varphi(s,z)\widetilde{N}(dz,ds)]$ is zero if $\int_0^t \int \mathbb{E}|\varphi(s,z)|^2\nu(dz)ds < \infty$ (see Gikhman and Skorokhod [53], Section II.2.5). This condition is similar to the well known Itô isometry for Brownian motion, and occurs because the stochastic integral with respect to the Poisson martingale measure $\widetilde{N}$ is defined in an L^2-space.

This stochastic integral can also be defined on the space of adapted functions φ with

$$\|\varphi\|_{\nu,2,1} := \int_0^t \int_{|z|<1} \mathbb{E}|\varphi(s,z)|^2\nu(dz)ds + \int_0^t \int_{|z|\geq 1} \mathbb{E}|\varphi(s,z)|\nu(dz)ds < \infty;$$

this definition is a true extension of the L^2-integral if $\nu(\{|z| \geq 1\}) < \infty$. The construction follows along the same lines as in Gikhman and Skorokhod [53], except that the isometry $\mathbb{E}\left|\int_0^t \int \varphi(s,z)\widetilde{N}(dz,ds)\right|^2 = \int_0^t \int \mathbb{E}|\varphi(s,z)|^2\nu(dz)ds$ is replaced by an inequality corresponding to the choice of $\|\cdot\|_{\nu,2,1}$,

$$\mathbb{E}\left[\left|\int_0^t \int_{|z|<1} \varphi(s,z)\widetilde{N}(dz,ds)\right|^2 + \left|\int_0^t \int_{|z|\geq 1} \varphi(s,z)\widetilde{N}(dz,ds)\right|\right] \leq \|\varphi\|_{\nu,2,1}. \tag{A.2}$$

The inequality is first proved for simple functions; these simple functions are adapted and have (compact) support in $\mathbb{R}^k \setminus \{0\}$. The stochastic integral for arbitrary adapted functions φ with $\|\varphi\|_{\nu,2,1} < \infty$ is defined as usual as limit of stochastic integrals over simple functions approximating φ in the $\|\cdot\|_{\nu,2,1}$ norm.

As $\mathbb{E}[\int_0^t \int \varphi(s,z)\widetilde{N}(dz,ds)] = 0$ holds for simple functions φ, it holds also for all adapted φ with $\|\varphi\|_{\nu,2,1} < \infty$. Thus for

$$M_s := \int_0^s \int \varphi(s,z)\widetilde{N}(dz,ds),$$

$(M_s)_{0\le s\le t}$ is a martingale if $\|\varphi\|_{\nu,2,1} < \infty$.

A.1.2 A localized version of Dynkin's formula

Our objective is to prove a localized version of Dynkin's formula, where the upper integration bound in (A.1) is some stopping time $\tau_\rho := \inf\{t > 0 : |X_t - X_0| \ge \rho\}$ with $\rho > 0$ arbitrarily small.

Our setting is similar to that of Chapter 2. Consider a process X evolving according to the stochastic differential equation (SDE)

$$dX_t = \mu(t, X_{t-})\,dt + \sigma(t, X_{t-})\,dW_t + \int \ell(t, X_{t-}, z)\,\overline{N}(dz, dt), \tag{A.3}$$

for a standard Brownian motion W and a compensated Poisson random measure $\overline{N}(dz,dt) = N(dz,dt) - 1_{|z|<1}\nu(dz)dt$. We assume in the following that there is a unique solution to (A.3). The generator $\mathcal{L}$ associated with this SDE is given by

$$\begin{aligned}\mathcal{L}u(t,x) = \frac{1}{2}tr\left(\sigma(t,x)\sigma^T(t,x)D_x^2u(t,x)\right) + \langle\mu(t,x), \nabla_x u(t,x)\rangle \\ + \int u(t, x+\ell(t,x,z)) - u(t,x) - \langle\nabla_x u(t,x), \ell(t,x,z)\rangle 1_{|z|<1}\,\nu(dz).\end{aligned}$$

We take the Lévy measure ν as given. As for all Lévy measures, $\int(|z|^2 \wedge 1)\,\nu(dz) < \infty$. Let $p^* > 0$ be a number such that $\int_{|z|\ge 1}|z|^{p^*}\,\nu(dz) < \infty$, and let $q^* \ge 0$ be a number such that $\int_{|z|<1}|z|^{q^*}\,\nu(dz) < \infty$ (think of p^* as the *largest* and q^* the *smallest* such number, even if it does not exist). We assume for all $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d$:

(A1) There are $a, b > 0$ such that $ab \le p^*$ and $|\ell(t,x,\beta,z)| \le C_{t,x}(1+|z|^a)$ on $|z| \ge 1$, $|u(t,z)| \le C(1+|z|^b)$. Furthermore, $|\ell(t,x,\beta,z)| \le C_{t,x}|z|^{q^*/2}$ on $|z| < 1$.

Theorem A.1.1. *Let Assumption (A1) be satisfied. Furthermore, assume that $u \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^d)$ and that σ is locally bounded. Then for any starting point $X_t = x$, there is a $\rho > 0$ such that $\mathcal{L}u$ is well-defined in $[t, t+\rho) \times B(x,\rho)$, and*

$$\mathbb{E}^{(t,x)}[u(\tau_\rho, X_{\tau_\rho})] = u(t,x) + \mathbb{E}^{(t,x)}\left[\int_t^{\tau_\rho} u_t(s, X_s) + \mathcal{L}u(s, X_s)ds\right]$$

for $\tau_\rho := \inf\{s \ge t : X_s^{t,x} \notin B(x,\rho)\} \wedge t + \rho$.

Note that all local properties and also the theorem extend to all compacts, by arguments given at the end of Chapter 1.

Proof: An easy estimation based on (A1), together with Taylor's formula for $\{|z| < 1\}$ shows the local well-definedness of $\mathcal{L}u$; see also the corresponding discussion in §2.2. Again by (A1) and results of §A.1.1, the process $(\int_t^s \int \ell(r, X_{r-}, z)\widetilde{N}(dz, ds))_{s\geq t}$ is a local martingale with localizing sequence $(\tau_\rho)_{\rho>0}$.

Because any local martingale is a semimartingale, we can apply Ito's formula to $u(\tau_\rho, X_{\tau_\rho})$ (see Protter [101] Th. II.33; compare also Gikhman and Skorokhod [53], Section II.2.6) and obtain

$$\begin{aligned} u(\tau_\rho, X_{\tau_\rho}) = u(t,x) &+ \int_t^{\tau_\rho} u_t(s, X_s) + \mathcal{L}u(s, X_s)ds + \int_t^{\tau_\rho} \langle \nabla_x u(s, X_s), \sigma(s, X_s)\rangle dW_s \\ &+ \int_t^{\tau_\rho} \int [u(s, X_{s-} + \ell(s, X_{s-}, z)) - u(s, X_{s-})]\widetilde{N}(dz, ds), \end{aligned}$$

where $\widetilde{N}(dz, ds) = N(dz, ds) - \nu(dz)ds$ is the fully compensated Poisson random measure. The remaining task is to prove that the integrals with respect to W and $\widetilde{N}$ disappear when taking expectation, for ρ small enough. For the Brownian motion integral, this is the case if $\mathbb{E}[\int_t^{\tau_\rho} |\langle \nabla_x u, \sigma\rangle|^2 ds] < \infty$, which holds trivially because σ and $\nabla_x u$ are locally bounded.

After appropriate localization with τ_ρ, the discussion in §A.1.1 applies and the expectation of the Poisson random measure integral is zero, because $\|u(s, X_{s-} + \ell(s, X_{s-}, z)) - u(s, X_{s-})\|_{\nu,2,1} < \infty$ locally by (A1) and similar arguments as above. □

A.2 Tools for the proof of Theorem 2.4.2

Lemma A.2.1. *Consider a process X^{β^n,t_n,x_n} following the SDE (2.4), started at (t_n, x_n), and controlled with a stochastic control β^n. Denote the first exit time for $\rho > 0$ by $\tau_n^\rho := \inf\{t \geq t_n : |X_t^{\beta^n,t_n,x_n} - x_n| \geq \rho\}$. Suppose that all conditions for existence and uniqueness are satisfied for X^{β^n}, that (E4) holds and that $(t_n, x_n) \to (t_0, x_0)$. Then for all $\delta > 0$ there is a $\hat{\varepsilon} > 0$ such that*

$$\limsup_{n\to\infty} \mathbb{P}(|\tau_n^\rho - t_0| < \hat{\varepsilon}) < \delta.$$

Proof: We want to prove that there is no subsequence in n such that $\mathbb{P}(|\tau_n^\rho - t_n| > \varepsilon) \to 0$ $(n \to \infty)$ for all $\varepsilon > 0$. Define

$$K_{n,\varepsilon} := \mathbb{P}(|\tau_n^\rho - t_n| > \varepsilon) = \mathbb{P}(\sup_{t_n\leq s\leq t_n+\varepsilon} |X_s^{\beta^n,t_n,x_n} - x_n| < \rho).$$

By stochastic continuity, we have for each fixed n that $K_{n,\varepsilon} \to 1$ for $\varepsilon \to 0$. This convergence is uniform in n for the following reasons: By (E4), we can find a constant $\hat{\beta}$ such that $X^{\hat{\beta},t_n,x_n}$ has a higher "variability" than X^{β^n,t_n,x_n}:

$$\mathbb{P}(\sup_{t_n\leq s\leq t_n+\varepsilon} |X_s^{\hat{\beta},t_n,x_n} - x_n| < \rho) \leq \mathbb{P}(\sup_{t_n\leq s\leq t_n+\varepsilon} |X_s^{\beta^n,t_n,x_n} - x_n| < \rho)$$

for all ε small enough and n large enough. Further, for n large enough such that $|x_n - x_0| < \rho/3$,

$$\begin{aligned} &\mathbb{P}(\sup_{t_n\leq s\leq t_n+\varepsilon} |X_s^{\hat{\beta},t_n,x_n} - x_n| < \rho) \\ &\qquad \geq \mathbb{P}(\sup_{t_n\leq s\leq t_n+\varepsilon} |X_s^{\hat{\beta},t_n,x_n} - X_s^{\hat{\beta},t_0,x_0}| < \rho/3 \ \wedge \sup_{t_0\leq s\leq t_0+\varepsilon} |X_s^{\hat{\beta},t_0,x_0} - x_0| < \rho/3) \end{aligned}$$

Finally, by stochastic continuity [in the initial condition] (cf. Gikhman and Skorokhod [53], p. 279), the right hand side converges to 1 for $\varepsilon \to 0$ (uniformly in n for n large enough). This means that for any subsequence $(K_{n_k,\varepsilon})_k$ converging in k for all $\varepsilon > 0$, there is an $\hat{\varepsilon}$ such that the limit must lie arbitrarily close to 1. $\square$

Lemma A.2.2. *Let $(X_t)_{t\geq 0}$ be a sequence of random variables, and let X_t converge to $X_0 = x \in \mathbb{R}^d$ in probability for $t \downarrow 0$. Then:*

(i) For all $\varepsilon > 0$, and for all sequences $t_n \to 0$,

$$\mathbb{P}(\sup_{0\leq s\leq t_n} |X_s - x| > \varepsilon) \to 0, \qquad n \to \infty$$

(ii) (i) holds true also for a sequence of positive random variables (τ_n) converging to 0 in probability (not necessarily independent of X).

(iii) (ii) holds true also in the setting and under the assumptions of Lemma A.2.1, i.e.,

$$\mathbb{P}(\sup_{t_n\leq s\leq \tau_n} |X_s^{\beta^n,t_n,x_n} - x_n| > \varepsilon) \to 0$$

for $(t_n, x_n) \to (t_0, x_0)$, random variables $\tau_n \geq t_n$, $\tau_n \to t_0$ in probability.

Proof: (i): By assumption, for all $\varepsilon, \delta > 0$ there is a set $U = [0, u]$ for $u > 0$ such that for all $s \in U$, $\mathbb{P}(|X_s - x| > \varepsilon) < \delta$. Fix now $\varepsilon, \delta > 0$. Dependent on ω, $\gamma > 0$ and $t > 0$, choose $\hat{t}(\omega, \gamma, t) \in [0, t]$ such that $|X_{\hat{t}(\omega,\gamma,t)} - x| \geq \sup_{0\leq s\leq t} |X_t(\omega) - x| - \gamma$.
Select now $\gamma := \varepsilon/2$. Then,

$$\mathbb{P}(\sup_{0\leq s\leq t} |X_s - x| > \varepsilon) \leq \mathbb{P}(|X_{\hat{t}(\omega,\gamma,t)} - x| > \varepsilon/2)$$

Let $U = [0, u]$ be such that $\mathbb{P}(|X_s - x| > \varepsilon/2) < \delta$ for all $s \in U$. Now choose $t := u$, so $\hat{t}(\omega, \gamma, t) \in [0, u]$, and thus $\mathbb{P}(\sup_{0\leq s\leq t} |X_s - x| > \varepsilon) < \delta$.

(ii): Set for a fixed $\varepsilon > 0$ $A_t := \{\omega : \sup_{0\leq s\leq t} |X_s - x| > \varepsilon\}$. We know that $\mathbb{P}(A_t) \to 0$ for $t \to 0$, or, equivalently, $1_{A_t} \to 0$ a.s. Now let $t > 0$ be fixed, and (τ_n) be a sequence of random variables converging to 0 in probability. Then, a.s.,

$$1_{A_t} = 1_{A_t} 1_{\{t<\tau_n\}} + 1_{A_t} 1_{\{t\geq\tau_n\}} \geq 1_{A_t} 1_{\{t<\tau_n\}} + 1_{A_{\tau_n}} 1_{\{t\geq\tau_n\}}$$

because $1_{A_s} \leq 1_{A_t}$ for $s \leq t$. The convergence $n \to \infty$ shows that $1_{A_t} \geq \limsup_{n\to\infty} 1_{A_{\tau_n}}$. For $t \to 0$ we get:

$$0 \geq \limsup_{n\to\infty} 1_{A_{\tau_n}} \geq 0,$$

thus $1_{A_{\tau_n}} \to 0$ a.s. for $n \to \infty$.

(iii): Consider $A_t^n := \{\omega : \sup_{t_n\leq s\leq t} |X_s^{\beta_n,t_n,x_n} - x_n| > \varepsilon_1\}$ in the proof of (ii). Then it can be checked in the proof that also $\mathbb{P}(A_{\tau_n}^n) \to 1$ for $n \to \infty$, because by the proof of Lemma A.2.1, $\mathbb{P}(A_t^n) \to 1$ for $t \to t_0$, $t \geq t_n$, uniformly in n large. $\square$

A.3 Pricing in a Markov-switching economy

We shortly describe here how the price of infinite-maturity loans in a Markov-switching economy can be derived, assuming that all parameters are risk-neutral; this price was used in the discussion of §4.2.2. First we calculate the risk-neutral valuation formulas for a loan with maturity $T > 0$. If τ is the default time of the loan, then its price $p_i^T(0)$ if we start in $t = 0$ with economy state i is determined by:

$$\begin{aligned} p_i^T(0) &= \mathbb{E}^{(0,i)}\left[\int_0^{T\wedge\tau} e^{-\rho t} r_L dt + (1-\delta(M_\tau))e^{-\rho\tau}1_{\tau\le T} + e^{-\rho T}1_{\tau>T}\right] \\ &= \int_0^T \mathbb{E}^{(0,i)}[e^{-\int_0^t \lambda(M_s)ds}]e^{-\rho t} r_L dt + \int_0^T \mathbb{E}^{(0,i)}[(1-\delta(M_t))\lambda(M_t)e^{-\int_0^t \lambda(M_s)ds}]e^{-\rho t}dt \\ &\quad + \mathbb{E}^{(0,i)}[e^{-\int_0^t \lambda(M_s)ds}]e^{-\rho T} \end{aligned} \tag{A.4}$$

where the last equality was obtained by conditioning on the filtration generated by the economy process (see, e.g., Th. 9.23 in McNeil et al. [81]), and interchanging integration and expectation. We see that we have to determine for some function f and $T > t$ the expectation $v_i(t,x) := \mathbb{E}[f(M_T)e^{-X_T}|M_t = i, X_t = x]$ for $dX_s = \lambda(M_s)ds$. v is the unique solution to the parabolic PDE

$$v_t + \begin{pmatrix} \lambda(0) & 0 \\ 0 & \lambda(1) \end{pmatrix} v_x + \begin{pmatrix} -\lambda_{01} & \lambda_{01} \\ \lambda_{10} & -\lambda_{10} \end{pmatrix} v = 0, \qquad v(T,x) = \begin{pmatrix} f(0) \\ f(1) \end{pmatrix} e^{-x} \tag{A.5}$$

on $(0,T)\times\mathbb{R}$. Because we know that $v(t,x) = e^{-x}v(t,0)$ and thus $v_x = -v$, we have to solve the standard ODE

$$v' = A_\lambda v, \qquad\qquad v(T) = \boldsymbol{f}e^{-x}$$

for $A_\lambda = \begin{pmatrix} \lambda(0)+\lambda_{01} & -\lambda_{01} \\ -\lambda_{10} & \lambda(1)+\lambda_{10} \end{pmatrix}$ and $\boldsymbol{f} = \begin{pmatrix} f(0) \\ f(1) \end{pmatrix}$, which has the general solution $v(t,x) = \exp(-A_\lambda(T-t))\boldsymbol{f}e^{-x}$. Coming back to our original problem (A.4), by formal integration of the matrix exponential, we obtain

$$p^T(0) = A_{\lambda,\rho}^{-1}(I - \exp(-A_{\lambda,\rho}T))\left(r_L\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} (1-\delta(0))\lambda(0) \\ (1-\delta(1))\lambda(1) \end{pmatrix}\right) + \exp(-A_{\lambda,\rho}T), \tag{A.6}$$

where $A_{\lambda,\rho} := A_\lambda + \rho I$ for the unity matrix I. The corresponding formula for an infinite-maturity loan can be obtained by $T \to \infty$:

$$p^\infty(0) = A_{\lambda,\rho}^{-1}\left(r_L\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} (1-\delta(0))\lambda(0) \\ (1-\delta(1))\lambda(1) \end{pmatrix}\right). \tag{A.7}$$

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Daten zum Autor

Name:	Roland Seydel
Geburtsdatum:	12. November 1980 in München
11/1999 - 10/2004	Studium der Finanz- und Wirtschaftsmathematik Technische Universität München
01/2005 - 10/2007	Consultant für Risiko-Management und Bewertung d-fine GmbH, Frankfurt
11/2007 - 09/2009	Doktorand am Max-Planck-Institut für Mathematik in den Naturwissenschaften, Leipzig

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Leipzig, den 10. September 2009

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(Roland Seydel)